

## Physics 127c: Statistical Mechanics

### Fermi Liquid Theory: Principles

Landau developed the idea of quasiparticle excitations in the context of interacting Fermi systems. His theory is known as *Fermi liquid theory*. He introduced the idea phenomenologically, and later Abrikosov and Kalatnikov gave a formal derivation using diagrammatic perturbation theory to all orders.

Landau suggested describing the excited states of the interacting system as in one-to-one correspondence with the excited states of the noninteracting system, through “switching on” the pair interactions. The interactions conserve the total particle number, spin, and momentum. Starting with a noninteracting system with one particle added in state  $\mathbf{p}, \sigma$  to the ground state Fermi sea, and switching on the interactions, so that the particle becomes “dressed” by its interaction with the other particles, gives a state with characteristics of a particle in an excited state with definite momentum  $\mathbf{p}$ , spin state  $\sigma$ , and adding one to the particle count. The energy of course is not preserved because the Hamiltonian is changed. In addition the state given by this switch-on process will eventually decay into a collection of more complicated states (e.g. by exciting particle-hole pairs out of the Fermi sea) so that there is a finite lifetime. Thus the process gives a state with simple quantum numbers  $\mathbf{p}, \sigma, \mathbf{N}$ , and counting, because of the one-to-one correspondence with the noninteracting system, but it is not a true eigenstate of the interacting Hamiltonian. It is called a *quasiparticle* or *quasi-excitation*.

In the noninteracting system particles can only be added for  $p > p_F$ , and so this gives quasiparticle excitation with  $p > p_F$ . (Remember,  $p_F$  is not changed by interactions.) For  $p < p_F$ , no particles can be added to the noninteracting system, but a particle can be removed from  $\mathbf{p}, \sigma$  to form an excited state (of the  $N - 1$  particle system). Switching on the interaction now gives a quasihole state with momentum  $-\mathbf{p}, -\sigma$ . We can account for both types of excitations in terms of a change in occupation number  $\delta n_{\mathbf{p},\sigma}$  which is  $+1$  for the added particle/quasiparticle for  $p > p_F$ , and  $-1$  for the removed particle or hole/quasihole for  $p < p_F$ . In this notation we are using the filled Fermi sea as a reference for the quasiparticles.

The idea of switching on the interaction to define the quasi-excitations only makes sense if an appropriate switching rate  $\tau_s^{-1}$  can be found. This has to be slow enough that perturbations in the energy  $\sim \hbar/\tau_s$  are small compared to the energy scale of interest. This is of order  $p^2/2m - \varepsilon_F \sim v_F(p - p_F)$  with  $v_F \sim p_F/m$  the Fermi velocity. On the other hand  $\tau_s$  must be shorter than the lifetime of the quasiparticle, otherwise it will decay away during its birth. The lowest order decay process is scattering a particle out of the Fermi sea. Applying the Fermi Golden rule shows that the decay rate of a quasiparticle of momentum  $\mathbf{p}$  will vary proportionate to  $(p - p_F)^2$  for  $p$  near  $p_F$ , since by energy conservation and the Pauli exclusion principle the two particles must scatter into a narrow band of states of width about  $(p - p_F)$  near the Fermi surface. Thus the quasiparticle is well defined only for  $p \rightarrow p_F$ —typically we might guess for  $|p - p_F| \ll p_F$ , although the estimate must also depend on the strength of the interactions.

For a small number of quasiexcitations the energy relative to the ground state is given by superposition

$$E - E_0 = \sum_{\mathbf{p},\sigma} \varepsilon_p \delta n_{\mathbf{p},\sigma} + O(\delta n^2) \quad (1)$$

where  $\varepsilon_p = \delta E/\delta n_{\mathbf{p},\sigma}$  is the single quasiparticle energy, which will depend just on  $|p|$  for a rotationally invariant spin 1/2 system<sup>1</sup>. In general  $\varepsilon_p \neq p^2/2m$ .

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<sup>1</sup>Lifshitz and Pitaevskii give an argument for why there are no “spin-orbit interaction” terms for spin-1/2 particles. They also say that even if the bare Fermions are higher spin, the quasiparticles have spin 1/2.

## Propagator Approach

Landau was brilliant enough to correctly construct the picture introduced above, but there are certainly some mysteries left for the rest of us. A more pedestrian approach is to try to add a particle to the *interacting* ground state, and “see what happens”. This is captured by the propagator or Green function introduced in [Lecture 2](#), which in Fourier representation is

$$G(\mathbf{k}, t - t') = \begin{cases} -i \langle \psi_G | a_{\mathbf{k}\sigma}(t) a_{\mathbf{k}\sigma}^+(t') | \psi_G \rangle & t > t' \\ +i \langle \psi_G | a_{\mathbf{k}\sigma}^+(t') a_{\mathbf{k}\sigma}(t) | \psi_G \rangle & t \leq t' \end{cases} . \quad (2)$$

(This is known as the *time ordered product* of operators.) I use  $|\psi_G\rangle$  as the notation for the ground state of the *interacting* system, and will use  $|\psi_0\rangle$  for the ground state of the noninteracting system.

The time dependence of the operators is the Heisenberg dependence with the full Hamiltonian

$$a_{\mathbf{k}\sigma}(t) = e^{iHt/\hbar} a_{\mathbf{k}\sigma} e^{-iHt/\hbar} \quad (3)$$

with  $a_{\mathbf{k}\sigma}$  the Schrodinger version of the operator. Thus for  $t > t'$  we can write  $G$  as

$$G(\mathbf{k}, t - t') = -i \langle \psi_G | e^{iE_G t/\hbar} a_{\mathbf{k}\sigma} e^{-iH(t-t')/\hbar} a_{\mathbf{k}\sigma}^+ e^{-iE_G t'/\hbar} | \psi_G \rangle \quad (4)$$

which tells us about the evolution for the time  $t - t'$  of the state  $a_{\mathbf{k}\sigma}^+ |\psi_G(t')\rangle$  (the state given by adding a particle to  $\mathbf{k}, \sigma$  at time  $t'$ ), and in particular what is the overlap with the state  $a_{\mathbf{k}\sigma}^+ |\psi_G(t)\rangle$  after this time. For  $t < t'$  we learn about the propagation of  $a_{\mathbf{k}\sigma} |\psi_G(t)\rangle$ , the state with a particle removed.

## Noninteracting System

Let's first look at the noninteracting system, defined by the Hamiltonian

$$H_0 = \sum_{\mathbf{k}, \sigma} \varepsilon_k a_{\mathbf{k}, \sigma}^+ a_{\mathbf{k}, \sigma} . \quad (5)$$

with  $\varepsilon_k = \hbar^2 k^2 / 2m$ . The Heisenberg equation of motion

$$i\hbar \frac{da_{\mathbf{k}, \sigma}}{dt} = [a_{\mathbf{k}, \sigma}, H_0] \quad (6)$$

gives

$$a_{\mathbf{k}, \sigma}(t) = a_{\mathbf{k}, \sigma} e^{-i\varepsilon_k t/\hbar} , \quad (7a)$$

$$a_{\mathbf{k}, \sigma}^+(t) = a_{\mathbf{k}, \sigma}^+ e^{i\varepsilon_k t/\hbar} , \quad (7b)$$

so that the noninteracting propagator is

$$G_0(\mathbf{k}, \tau) = \begin{cases} -i \langle \psi_0 | a_{\mathbf{k}\sigma} a_{\mathbf{k}\sigma}^+ | \psi_0 \rangle e^{-i\varepsilon_k \tau/\hbar} & \tau > 0 \\ +i \langle \psi_0 | a_{\mathbf{k}\sigma}^+ a_{\mathbf{k}\sigma} | \psi_0 \rangle e^{-i\varepsilon_k \tau/\hbar} & \tau < 0 \end{cases} , \quad (8)$$

$$= e^{-i\varepsilon_k \tau/\hbar} \begin{cases} -i(1 - f_k) e^{-i\varepsilon_k \tau/\hbar} & \tau > 0 \\ i f_k e^{-i\varepsilon_k \tau/\hbar} & \tau < 0 \end{cases} , \quad (9)$$

with  $f_k = f(\varepsilon_k)$  the Fermi step function ( $f_k = 1$  for  $\varepsilon_k < \varepsilon_F$ ).

Now introduce the frequency-Fourier transform

$$G(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} G(\mathbf{k}, \tau) e^{i\omega\tau}, \quad (10a)$$

$$G(\mathbf{k}, \tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G(\mathbf{k}, \omega) e^{-i\omega\tau}. \quad (10b)$$

Imagine performing the inverse transform  $G(\mathbf{k}, \omega) \rightarrow G(\mathbf{k}, \tau)$  by contour integration. The integration is along the real axis. For  $\tau > 0$  we can close the contour at  $\infty$  in the lower half plane, since  $e^{-i\omega\tau} \rightarrow 0$  here. The integral is given in terms of the residues of the poles in the lower half plane. Similarly for  $\tau < 0$  the integral is given in terms of the residues of the poles in the upper half plane. The expression Eq. (9) is the inverse of

$$G_0(\mathbf{k}, \omega) = \frac{1}{\omega - \varepsilon_k/\hbar + i\eta_k}, \quad (11)$$

with  $\eta_k$  a positive infinitesimal for  $k > k_F$  and a negative infinitesimal for  $k < k_F$ . Thus the energy to add a particle is given by a pole in  $G_0(\mathbf{k}, \omega)$  slightly displaced from the real axis. The exact Fourier inverse gives a time dependence

$$G_0(\mathbf{k}, \tau) \propto e^{-i\varepsilon_k\tau/\hbar} e^{-|\eta_k|\tau}. \quad (12)$$

Then set  $\eta_k \rightarrow 0$ .

## Interacting System

Now return to the interacting system. We cannot expect to calculate a closed-form expression in general. Insight is gained from the *spectral representation*. This is generated by inserting a complete set of *exact* energy eigenstates  $\sum_m |\psi_m\rangle \langle\psi_m| = 1$  between the creation and annihilation operators in Eq. (2). For  $t > t'$  these are the eigenstates for the  $N + 1$  particle system  $|\psi_m^{(N+1)}\rangle$  with energies  $E_m^{(N+1)}$ ; for  $t < t'$  they are the states  $|\psi_m^{(N-1)}\rangle$  with energies  $E_m^{(N-1)}$ . Since momentum is a good quantum number, these states all have momentum  $\hbar\mathbf{k}$ . Now use

$$E_m^{(N+1)} - E_G^{(N)} = (E_m^{(N+1)} - E_G^{(N+1)}) + (E_G^{(N+1)} - E_G^{(N)}) \quad (13)$$

$$\simeq \varepsilon_m^{(N+1)} + \mu, \quad (14)$$

with  $\varepsilon_m^{(N+1)}$  the  $m$ th excitation energy of the  $N + 1$  particle system (necessarily positive), and  $\mu$  the chemical potential. Equation (2) is

$$G(\mathbf{k}, \tau) = \begin{cases} -i \sum_m |(a_{\mathbf{k},\sigma}^+)_{m,G}|^2 e^{-i\varepsilon_m^{(N+1)}\tau/\hbar} e^{-i\mu\tau/\hbar} & \tau > 0 \\ +i \sum_m |(a_{\mathbf{k},\sigma})_{m,G}|^2 e^{i\varepsilon_m^{(N-1)}\tau/\hbar} e^{-i\mu\tau/\hbar} & \tau \leq 0 \end{cases}, \quad (15)$$

with

$$|(a_{\mathbf{k},\sigma}^+)_{m,G}|^2 = \left| \langle \psi_m^{(N+1)} | a_{\mathbf{k}\sigma}^+ | \psi_G^{(N)} \rangle \right|^2, \quad (16a)$$

$$|(a_{\mathbf{k},\sigma})_{m,G}|^2 = \left| \langle \psi_m^{(N-1)} | a_{\mathbf{k}\sigma} | \psi_G^{(N)} \rangle \right|^2. \quad (16b)$$

The structure of the Fourier representation  $G(\mathbf{k}, \omega)$  is shown in Fig. 1. There are poles at the exact excitation energies

$$\hbar\omega = \begin{cases} \mu + \varepsilon_m^{(N+1)} - i\eta & \text{with residue } |(a_{\mathbf{k},\sigma}^+)_{m,G}|^2 \\ \mu - \varepsilon_m^{(N-1)} + i\eta & \text{with residue } |(a_{\mathbf{k},\sigma})_{m,G}|^2 \end{cases}, \quad (17)$$

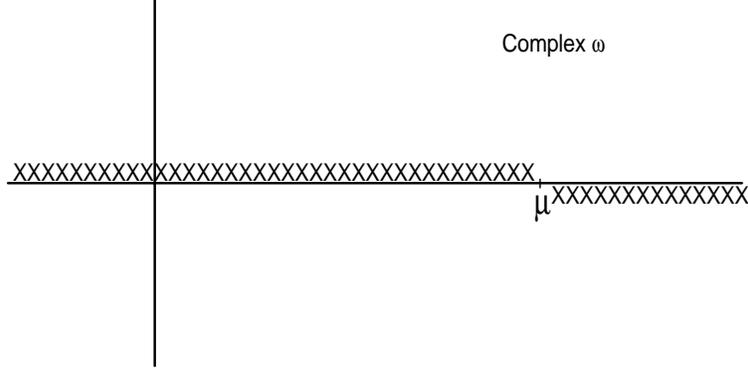


Figure 1: Analytic structure of full propagator  $G(\mathbf{k}, \omega)$  in the complex  $\omega$  plane. The X denote poles.

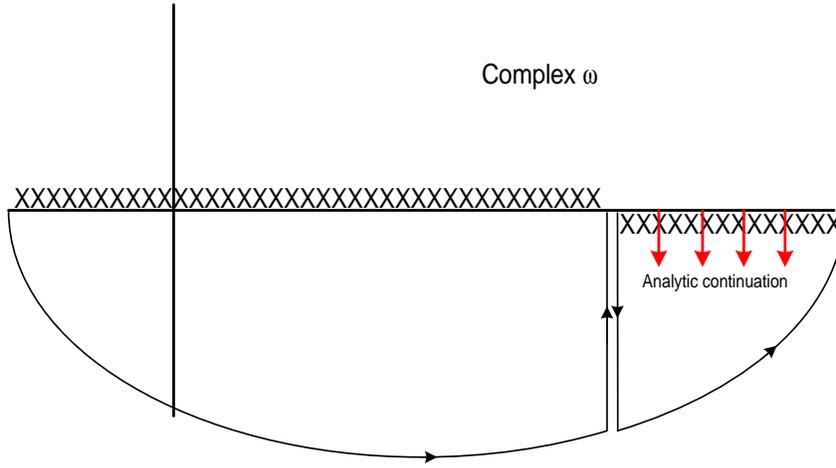


Figure 2: Analytic continuation in inverse Fourier transform.

with  $\eta$  a positive infinitesimal.

How does a quasiparticle appear in this picture? Let's focus on the added particle case  $k > k_F$ . In analogy with Eq. (12) we might expect

$$G(\mathbf{k}, \tau) \propto e^{-i\varepsilon_k \tau / \hbar} e^{-\gamma_k \tau} \quad (18)$$

with  $\gamma_k^{-1}$  a finite lifetime. This would give a pole in  $G(\mathbf{k}, \omega)$  for  $\omega > \mu$  and in the lower half plane, which is inconsistent with the general expression. To resolve this we must look at the contour integration more carefully, Fig. 2.

Consider the  $\tau > 0$  case when the contour for  $\text{Im } \omega \rightarrow -\infty$  gives zero. First we must recognize that the lifetime of the quasiparticle represented by the exponential decay of the propagator can only occur in the infinite system limit: if any finite system is "hit", as in particle addition, it will "ring" at the exact eigenstate frequencies. It is only in the infinite size limit that there is no recurrence, and the incoherence of the infinity of frequencies yields exponential decay. In the infinite size limit the poles become dense, and form a *branch cut* that cuts the complex  $\omega$  plane into two.

Split the integral along the real axis in the Fourier inverse into two pieces

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{\mu} + \int_{\mu}^{\infty}. \quad (19)$$

The first integral can be safely evaluated by displacing the contour to  $\infty$  in the lower half plane without encountering poles, and so reduces to just the contribution from the vertical portion from  $\mu - i\infty$  to  $\mu$ . We cannot do this for the second integral, since the real axis is cut off from the lower half plane by the branch cut. Instead we must define the analytic continuation of  $G(\mathbf{k}, \omega)$  from the real axis onto a second Riemann sheet in the lower half plane. *The quasiparticle appears as a pole in the analytic continuation of  $G(\mathbf{k}, \omega)$ .* A pole at  $\varepsilon_{\mathbf{k}} - i\gamma_{\mathbf{k}}$  will give a time dependence as in Eq. (18). It can be shown that for  $\gamma_{\mathbf{k}}$  sufficiently small, the contributions from the vertical portions of the contours in Fig. 2 give corrections to this leading order result.

Similarly the hole excitations are given by poles in the analytic continuation from the real axis  $\omega < \mu$  into the *upper* half plane, and contribute to  $G(\mathbf{k}, \tau)$  for  $\tau < 0$ .

If  $\gamma_{\mathbf{k}}$  is small, the value of  $G(\mathbf{k}, \omega)$  in the vicinity of real  $\omega \simeq \varepsilon_{\mathbf{k}}$  will become large, and this region will dominate the integral giving  $G(\mathbf{k}, t)$ . This corresponds to a clustering of exact eigenstates and/or a large values of their weights  $|(a_{\mathbf{k},\sigma}^+)_{m,G}|^2$  in this region. The quasiparticle picture captures all of this in terms of the single pole.

## Further Reading

*Statistical Mechanics, part 2* of the *Landau and Lifshitz* Theoretical Physics series (this volume is actually by *Lifshitz and Pitaevskii*) gives a nice, if typically terse, account: §1 discusses the phenomenological approach. A full discussion of the diagrammatic derivation (which certainly goes beyond the level of the present course) is only found in advanced Russian textbooks: the most famous is *Methods of Quantum Field Theory in Statistical Physics* by *Abrikosov, Gorkov, and Dzyaloshinskii*. It is also discussed in §14-20 of *Lifshitz and Pitaevskii*. For some reason, the standard American books on these methods (e.g. the ones by *Fetter and Walecka* or by *Mahan*) do not include the topic, to my mind one of the great triumphs of this formal approach.

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