Chapter 5

Two dimensional maps

One dimensional maps can only represent a truncated description of most physical dynamical systems, since the interesting maps—the ones with stretching and folding—are necessarily non-invertible and cannot be iterated in the reverse direction. On the other hand the dynamical equations of most physical systems can be integrated backwards in time to find the unique point in phase space that is the preimage of any point at the present time. *Two* dimensional maps

$$x_{n+1}^{(i)} = F_i\left(\left\{x_n^{(j)}\right\}\right) \qquad i, j = 1, 2 \tag{5.1}$$

can be a faithful representation of a smooth flow in a three dimensional phase space (e.g. as a Poincaré section). It is hard in general to construct (except numerically) the 2d map corresponding to a particular set of differential equations, and equally to reconstruct the smooth flow given a map. However, 2d maps that are invertible, but nevertheless show stretching and folding allowing positive Lyapunov exponents, are useful models of chaotic systems. Here we introduce four examples of two dimensional maps that have been discussed in various contexts. Further two dimensional maps will be introduced later in the investigation of quasiperiodic flows and their breakdown to chaos (chapters 18 and 21).

5.1 Henon Map

A particularly simple example of a 2-dimensional map is the Henon map [1]. The map iterates the point (x_n, y_n) via the equations

$$\begin{aligned} x_{n+1} &= y_n + 1 - a x_n^2 \\ y_{n+1} &= b x_n \end{aligned} \quad . \tag{5.2}$$

The proportional rate of expansion of an area of initial conditions is given by the Jacobean

$$J = \left| \det \frac{\partial F_i}{\partial x_n^{(j)}} \right| = \left| \det \begin{bmatrix} -2ax_n & 1 \\ b & 0 \end{bmatrix} \right| = |b| \tag{5.3}$$

so that for $b = 1$ the map is *area preserving* and for $b < 1$ the map is *dissipative*, with areas in phase space contracting at a constant proportionate rate. The latter is the regime of dissipative chaos. There is a useful connection with the one dimensional quadratic map in the strongly dissipative limit. We can rewrite the iteration process as

$$x_{n+1} = 1 - ax_n^2 - bx_{n-1} \tag{5.4}$$

so that in the limit $b \to 0$ we have approximately

$$x_{n+1} \simeq 1 - ax_n^2 \tag{5.5}$$

i.e. the quadratic one dimensional map. Since $y_{n+1} \propto x_n$ the plot of (x_n, y_n) reproduces the map function in this limit.

For commonly chosen values $a = 1.4$, $b = 0.3$ the evolution shows complex time behavior and the plot of the points visited in the long time limit (the attractor) shows a rich and complicated structure, so that at least numerically it appears to show chaos. (Again this appears to be one of those situations where for nearby parameter values there are attracting periodic orbits, and it has not been proven that for these particular values the orbit is chaotic, rather than periodic with a very long period.) The structure can be investigated in demonstration 1. One difficulty in studying the properties of the Henon map is that not all initial conditions iterate to the attractor: outside of a finite "basin of attraction" the iterations diverge to infinity.

5.2 Bakers' Map

A simple two dimensional map known as the (generalized) baker's map is useful for illustrating many of the ideas for characterizing chaos.

The generalized bakers' map is defined as the transformation of the unit square

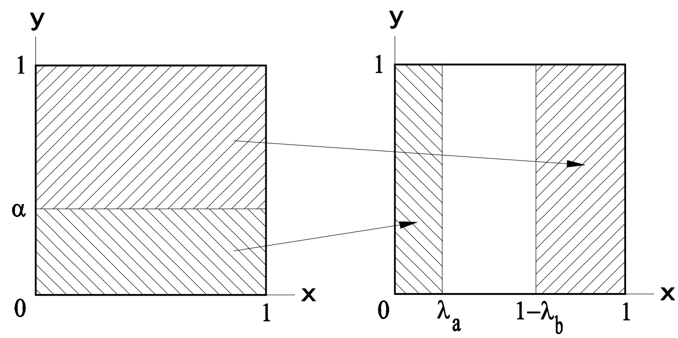

Figure 5.1: The Generalized Bakers' Map

$0 \le x \le 1, 0 \le y \le 1:$

$$\begin{aligned} x_{n+1} &= \begin{cases} \lambda_a x_n & \text{if} \quad y_n < \alpha \\ (1-\lambda_b) + \lambda_b x_n & \text{if} \quad y_n > \alpha \end{cases} \\ y_{n+1} &= \begin{cases} y_n/\alpha & \text{if} \quad y_n < \alpha \\ (y_n - \alpha)/\beta & \text{if} \quad y_n > \alpha \end{cases} \end{aligned} \tag{5.6}$$

where $\beta = 1 - \alpha$ and $\lambda_a + \lambda_b \le 1$. This is shown pictorially in figure 5.1. Note that the unit square is mapped into two nonoverlapping vertical stripes within the square, one in $0 \le x \le \lambda_a$ and one in $1 - \lambda_b \le x \le 1$. For $\lambda_a + \lambda_b = 1$ the map is area preserving; for $\lambda_a + \lambda_b < 1$ the map is dissipative.

For the dissipative map you can convince yourself of the following: after n iterations we have $C^n_m = \frac{n!}{m!(n-m)!}$ copies of stripes of width $\lambda_a^m \lambda_b^{n-m}$; and if we define a uniform measure over the original unit square, the measure associated with each stripe labelled by m at the nth iteration is $\alpha^m \beta^{n-m}$ and the measure is uniform in the y direction. The Bakers' map is explored in demonstration 2.

The Bakers' map is an example of an important class of maps for which the attractor is *hyperbolic*, which we can define crudely as ones for which expanding and contracting directions can be defined at each point, which are continuous across the attractor and for which the expansion and contraction rates are bounded away from zero. (For the Bakers' map the expanding direction is the y direction and the contracting direction is the x direction, and the rates are uniform and set by α, β, λ_a and λ_b.) Hyperbolicty allows various properties to be proved mathematically. However most maps and flows that occur naturally are not hyperbolic. The Henon map for example is not hyperbolic.

5.3 The Duffing Map

This map is reminiscent of the Duffing equation discussed in chapter 3. It is defined by the equations

$$\begin{aligned} x_{n+1} &= y_n \\ y_{n+1} &= -bx_n + ay_n - y_n^3 \end{aligned} . \tag{5.7}$$

The form of these equations can be motivated by using a finite difference approximation to evaluate the derivatives in the Duffing equation using times $t_n = nh$ with h a time step

$$\left.\frac{dx}{dt}\right|_{t=t_{n+1}} \simeq \frac{x_{n+2} - x_n}{2h} \tag{5.8}$$

$$\left.\frac{d^2x}{dt^2}\right|_{t=t_{n+1}} \simeq \frac{x_{n+2} - 2x_{n+1} + x_n}{h^2} \tag{5.9}$$

and then making the replacement $x(t = nh) \to x_n$ and $y_n = x_{n+1}$. Thus the combination $y_n - x_n$ is closest to the velocity variable of the differential equations.

5.4 Kaplan-Yorke Map

The Kaplan-Yorke map is defined by:

$$\begin{aligned} x_{n+1} &= ax_n \bmod 1 \\ y_{n+1} &= by_n + \cos(2\pi x_n) \end{aligned} . \tag{5.10}$$

It was used by Kaplan and Yorke [2] in their paper introducing the idea of "Lyapunov dimension" (see chapter 9).

The Duffing and Kaplan-Yorke maps are illustrated in demonstration 3.

Bibliography

[1] M. Henon, Comm. Math. Phys. **50**, 69 (1976)

[2] J.L. Kaplan and J.A. Yorke, Lecture Notes in Math **730**, 204 (1979)