

Why Chaos?

Conditions for Chaos

Necessary conditions for a set of evolution equations (assumed to be first order differential equations) to show chaos are:

- There must be at least three variables and equations
- There must be some nonlinearity

Kirchoff's equations for the circuit, involving equations for the voltages V_1 across C_1 , and V_2 across C_2 , and the current I through the inductor L

$$\begin{aligned}C_1 \frac{dV_1}{dt} &= R^{-1} (V_2 - V_1) - g(V_1) \\C_2 \frac{dV_2}{dt} &= -R^{-1} (V_2 - V_1) + I \\L \frac{dI}{dt} &= -rI - V_2\end{aligned}$$

satisfy these requirements:

- The requirement of 3 equations is given by the inclusion of three reactive elements C_1 , C_2 , and L
- The nonlinearity in the circuit comes from the piecewise linear **effective negative resistance** $g(V)$ produced by the op-amp, diodes and associated resistors. Thus the *only* nonlinearity in the effective circuit is the kink in the $I - V$ characteristics $g(V)$.

Oscillations

The d.c. operating point I_0, V_0 is the intersection of the "load line" of slope $-1/R$ with $g(V)$. There are actually two possibilities, at positive and negative values. For suitable circuit values these stationary solutions are unstable to growing oscillations about the d.c. point. Since $g(V)$ is locally linear, the amplitude of the oscillations will continue to grow, until the voltage somewhere in the cycle reaches the kink in $g(V)$ at V_c . The d.c. resistance in the circuit then becomes positive over part of the cycle, saturating (if we are lucky!) the growth.

A typical observation will be oscillations where the voltage V_1 oscillates about one of the two d.c. operating points $\pm V_0$ with an amplitude large enough to swing the voltage past the kink at $\pm V_c$.

Chaos

It is hard to predict the effect of the non-linearity: Does the periodic orbit persist or does it break down to chaos? Which of the routes to chaos occurs? What is the nature of the chaotic dynamics?

Observation on the physical circuit or the simulation shows that the periodic orbit undergoes several period doubling bifurcations (I managed to see up to period 16 in the physical circuit), and then becomes noisy. At first the noise is weak (as expected for the period doubling route to chaos), perturbing the main oscillation about V_0 , but eventually, as parameters change to increase the nonlinearity, the dynamics begins to switch randomly back and forth between oscillations about V_0 and then $-V_0$. In this regime the chaotic dynamics is quite reminiscent of Lorenz chaos. If we rescale the variables the equations

$$\begin{aligned}\frac{dX}{dt} &= a(Y - X) - \bar{g}(X) \\ \frac{dY}{dt} &= \sigma[-a(Y - X) + Z] \\ \frac{dZ}{dt} &= -c(Y + \bar{r}Z)\end{aligned}$$

are quite reminiscent of the Lorenz equations, except that the product nonlinearities there are replaced by the piecewise linear function of a single variable here.