# Collective Effects

## in

# Equilibrium and Nonequilibrium Physics

Website: http://cncs.bnu.edu.cn/mccross/Course/

Caltech Mirror: http://haides.caltech.edu/BNU/

# Today's Lecture: Symmetry Aspects of Patterns

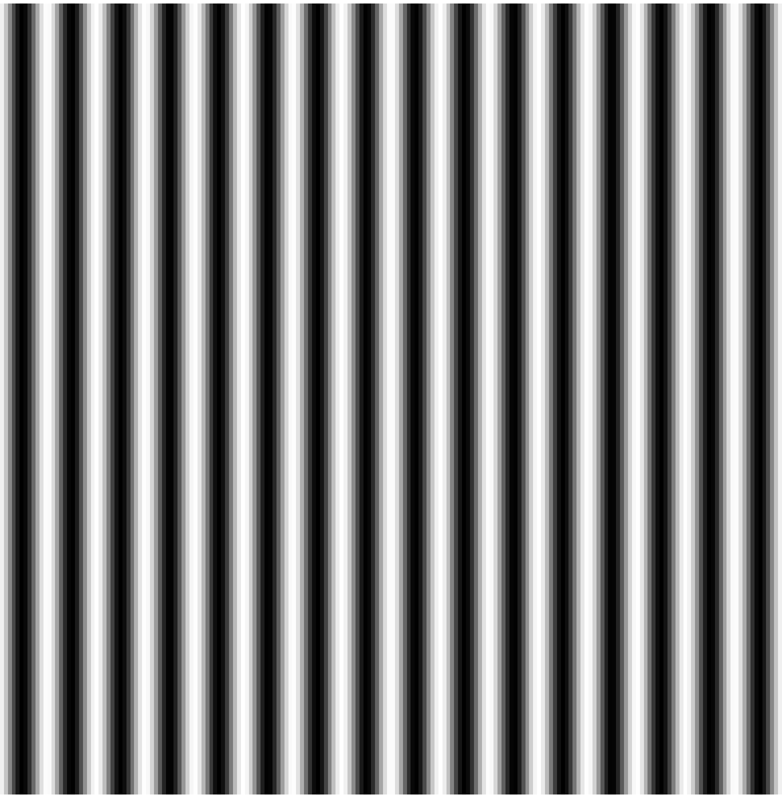
## Outline

- Broken symmetry, phase variable and Goldstone modes
- Phase equation near stationary instabilities
- Phase equations far from onset
- Topological defects
- Amplitude and phase equations for oscillatory instabilities

# Phase Dynamics of Stripe Patterns

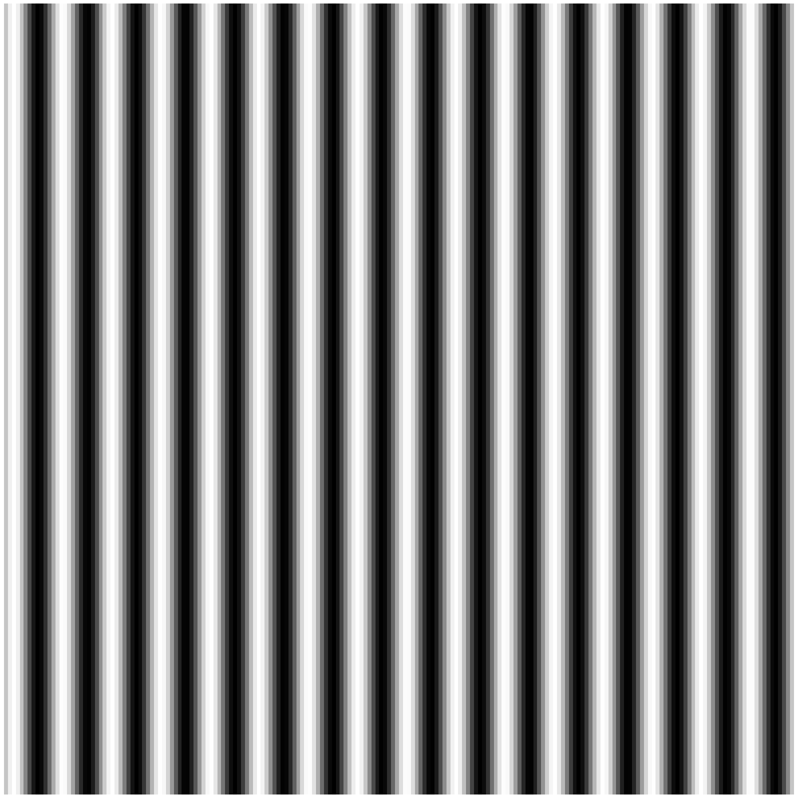
- The local structure of a stripe pattern is $\propto \cos(\mathbf{q}\cdot\mathbf{x}+\theta)+$ harmonics.
- A constant phase change is just a spatial shift of the pattern.
- A phase change that varies slowly in space (over a length $\eta^{-1}$, say, with $\eta$ small) will evolve slowly in time.
- For small enough $\eta$ the phase variation is slow compared with the relaxation of other degrees of freedom such as the magnitude or the internal structure, and a particularly simple description is obtained.
- The phase variable describes the symmetry properties of the system: the connection between symmetry and slow dynamics is known as Goldstone's theorem.
- Near onset $\theta$ is simply the phase of the complex amplitude, and an equation for the phase dynamics can be derived from the amplitude equation for $\eta \ll \varepsilon$ (Pomeau and Manneville, 1979)

# Effect of a Phase Change



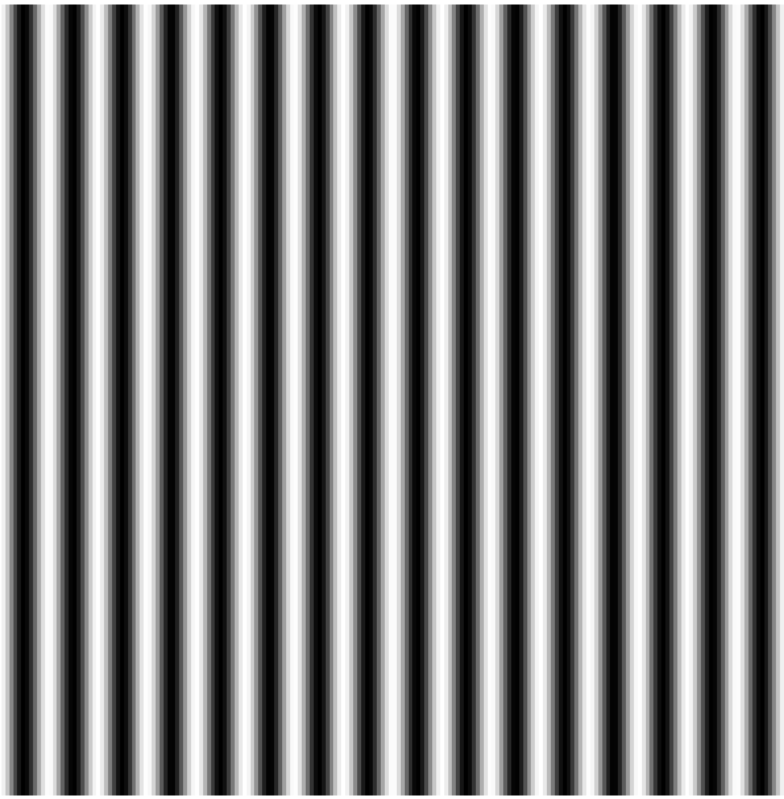
$$\cos(\mathbf{q}\cdot\mathbf{x}+\theta); \qquad \mathbf{q}=(1,0),\ \theta=0$$

# Effect of a Phase Change



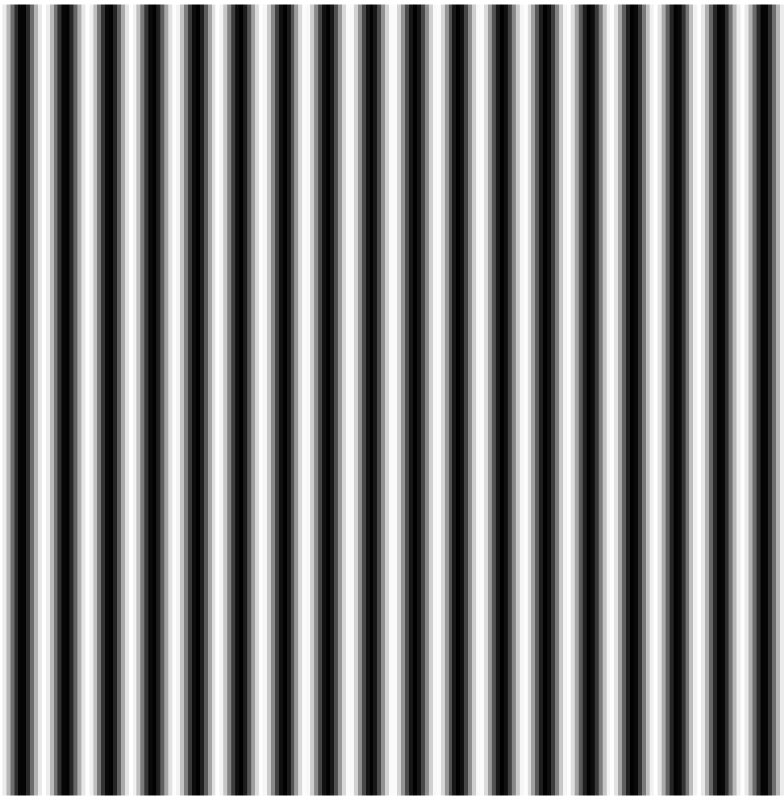
$$\cos(\mathbf{q}\cdot\mathbf{x}+\theta); \qquad \mathbf{q}=(1,0),\ \theta=1$$

# Effect of a Phase Change



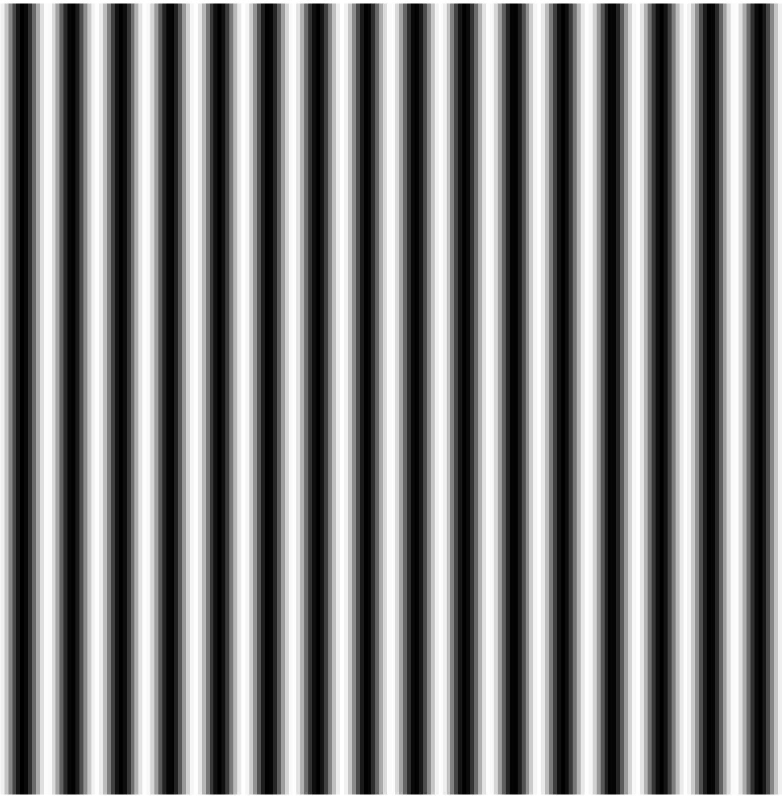
$$\cos(\mathbf{q}\cdot\mathbf{x}+\theta); \qquad \mathbf{q}=(1,0),\ \theta=0$$

# Effect of a Phase Change



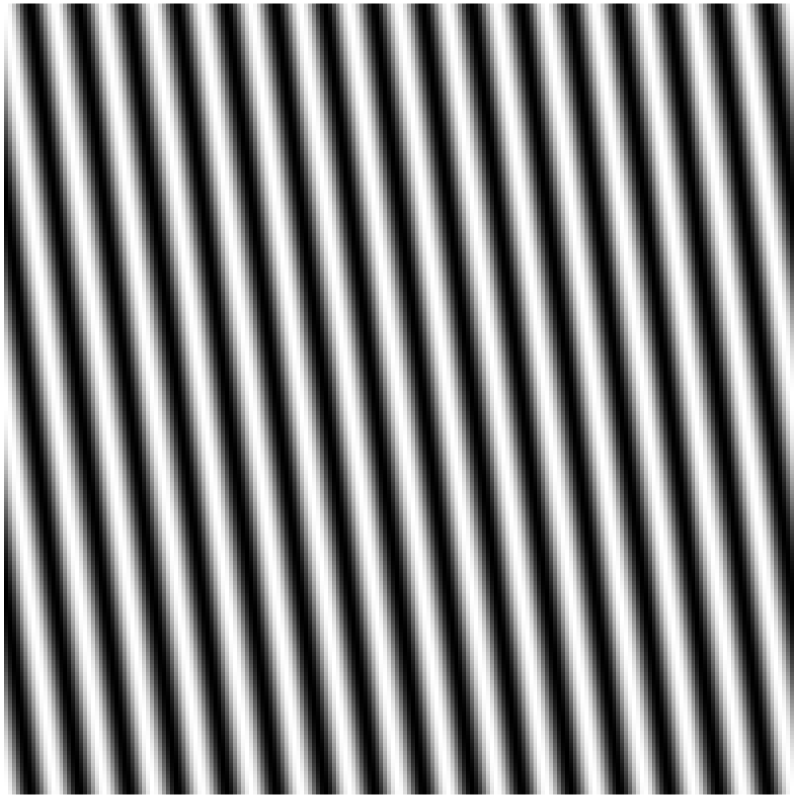
$$\cos(\mathbf{q}\cdot\mathbf{x}+\theta); \qquad \mathbf{q}=(1,0),\ \theta=\mathbf{Q}\cdot\mathbf{x},\ \mathbf{Q}=(1/8,0)$$

# Effect of a Phase Change



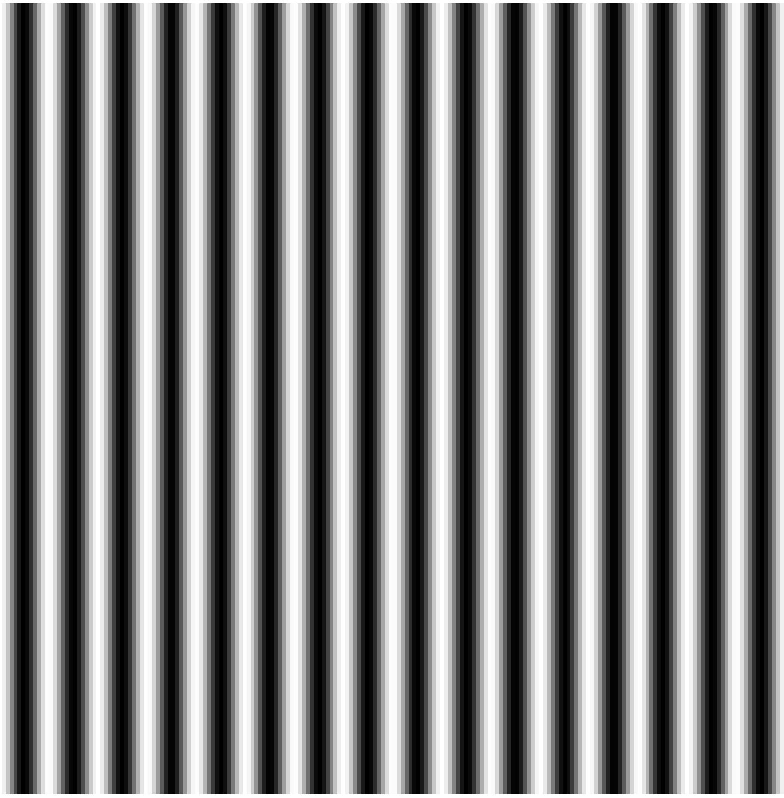
$$\cos(\mathbf{q}\cdot\mathbf{x}+\theta); \qquad \mathbf{q}=(1,0),\ \theta=0$$

# Effect of a Phase Change



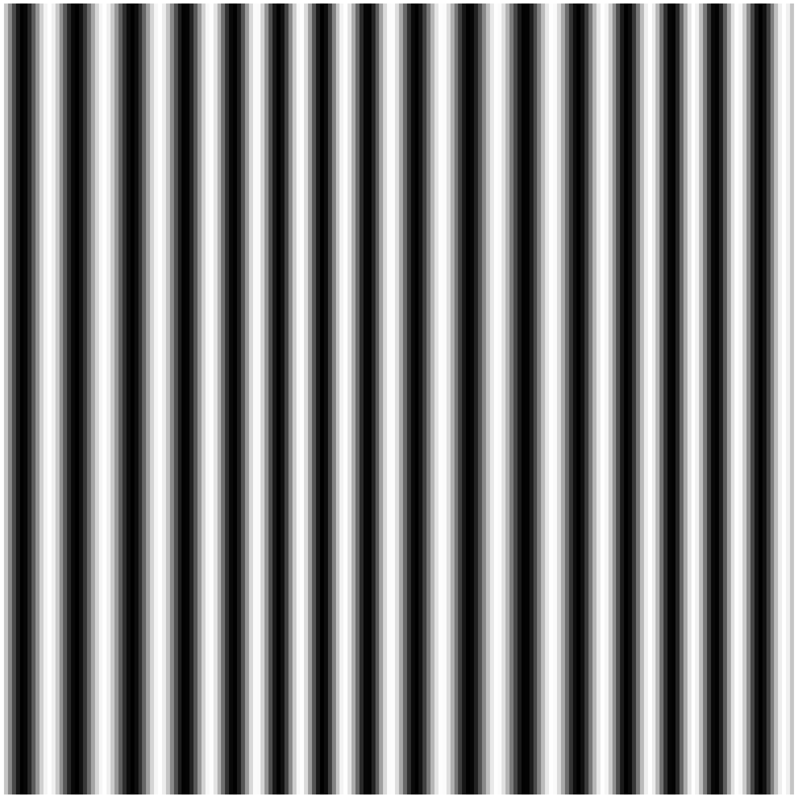
$$\cos(\mathbf{q}\cdot\mathbf{x}+\theta); \qquad \mathbf{q}=(1,0),\ \theta=\mathbf{Q}\cdot\mathbf{x},\ \mathbf{Q}=(0,1/8)$$

# Effect of a Phase Change



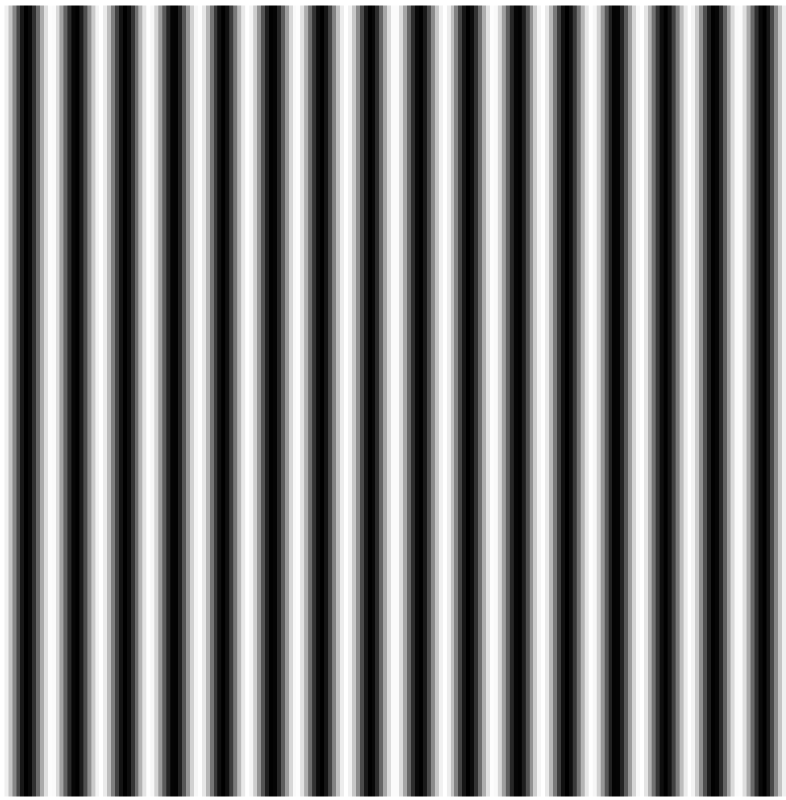
$$\cos(\mathbf{q}\cdot\mathbf{x}+\theta); \qquad \mathbf{q}=(1,0),\ \ \Phi=0$$

# Effect of a Phase Change



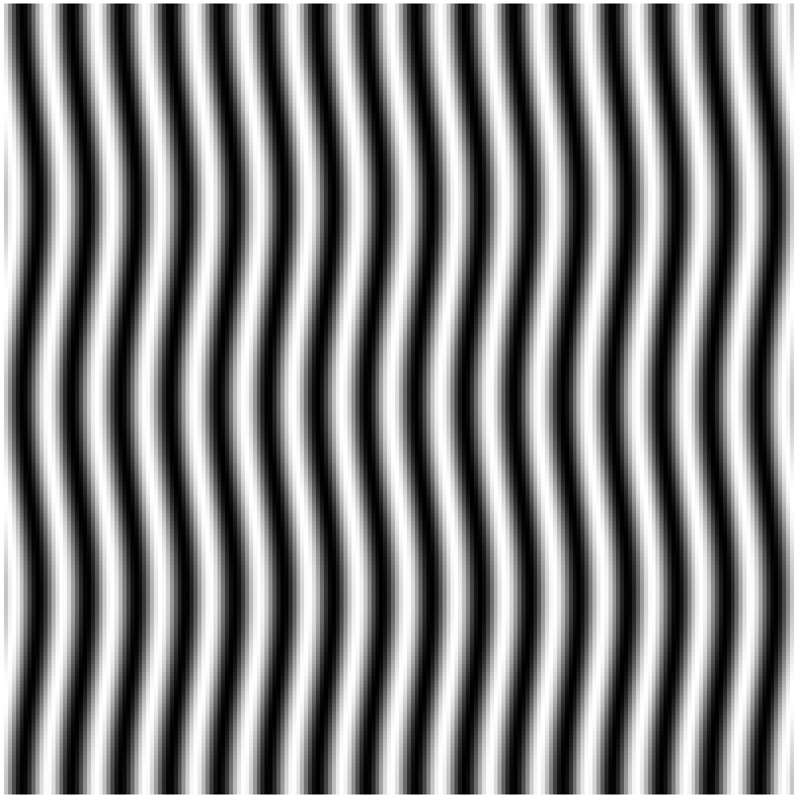
$$\cos(\mathbf{q}\cdot\mathbf{x}+\theta); \qquad \mathbf{q}=(1,0),\ \theta=\cos(\mathbf{Q}\cdot\mathbf{x}),\ \mathbf{Q}=(1/8,0)$$

# Effect of a Phase Change



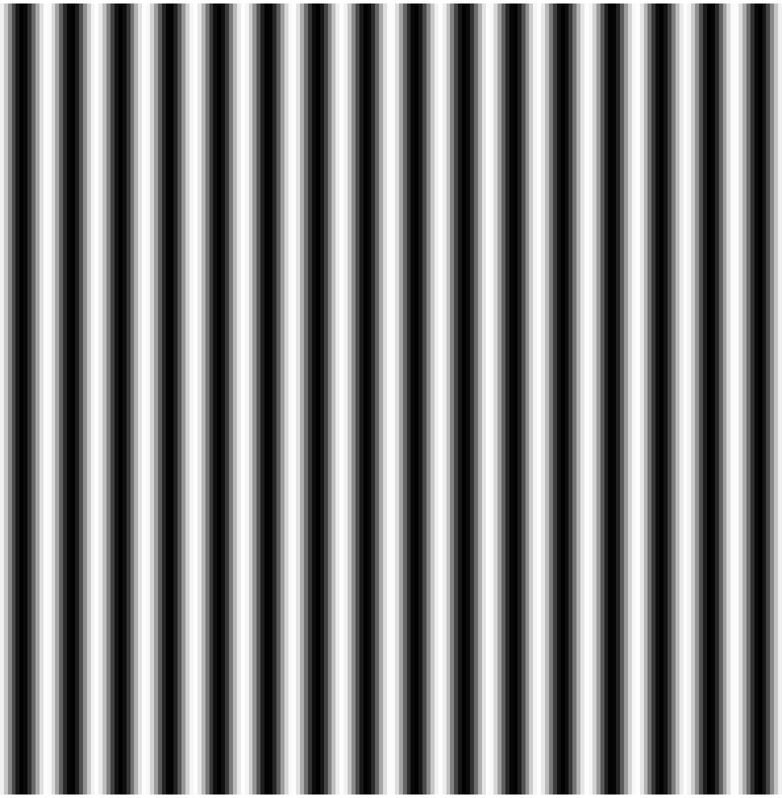
$$\cos(\mathbf{q}\cdot\mathbf{x}+\theta); \qquad \mathbf{q}=(1,0),\ \theta=0$$

# Effect of a Phase Change



$$\cos(\mathbf{q}\cdot\mathbf{x}+\theta); \qquad \mathbf{q}=(1,0),\ \theta=\cos(\mathbf{Q}\cdot\mathbf{x}),\ \mathbf{Q}=(0,1/8)$$

# Effect of a Phase Change



$$\cos(\mathbf{q}\cdot\mathbf{x}+\theta); \qquad \mathbf{q}=(1,0),\ \theta=0$$

## Phase Dynamics Near Onset

- Near threshold the phase reduces to the phase of the complex amplitude, and the phase equation can be derived by *adiabatically eliminating* the relatively fast dynamics of the magnitude.
- Basic assumption: we are looking at the dynamics driven by gradual spatial variations of the phase, i.e. derivatives of $\theta$ are small.
- For simplicity also assume that we are looking at small deviations from a straight stripe pattern, so that the phase perturbations may also be considered small.
- This leads to the *linear phase diffusion equation* first derived by Pomeau and Manneville (1979). We will consider the full nonlinear phase equation in the more general context away from threshold.

# Derivation of Phase Equation from Amplitude Equation I

Consider the (scaled) amplitude equation

$$\partial_T \bar{A} = \bar{A} + \left(\partial_X - \frac{i}{2}\partial_Y^2\right)^2 \bar{A} - \left|\bar{A}\right|^2 \bar{A}$$

Look at small perturbations about the state $\bar{A} = a_K e^{iKX}$ with $a_K^2 = 1 - K^2$,

$$\bar{A} = a e^{iKX} e^{i\theta}, \qquad a = a_K + \delta a$$

# Derivation of Phase Equation from Amplitude Equation I

Consider the (scaled) amplitude equation

$$\partial_T \bar{A} = \bar{A} + \left(\partial_X - \frac{i}{2}\partial_Y^2\right)^2 \bar{A} - \left|\bar{A}\right|^2 \bar{A}$$

Look at small perturbations about the state $\bar{A} = a_K e^{iKX}$ with $a_K^2 = 1 - K^2$,

$$\bar{A} = a e^{iKX} e^{i\theta}, \qquad a = a_K + \delta a$$

Expand in

- small phase perturbations $\theta$ and amplitude perturbations $\delta a$
- low order derivatives of $\theta$ (up to second order)

Then using

$$e^{-iKX} e^{-i\theta} \partial_T A = \partial_T a + i a \partial_T \theta,$$

the real part of the equation gives the dynamical equation for $a$, and the imaginary part of the equation gives the dynamical equation for $\theta$.

# Derivation of Phase Equation from Amplitude Equation II

## Real part

$$\partial_T \delta a = -2a_K^2 \delta a - 2K a_K \partial_X \theta + \partial_X^2 \delta a$$

For time variations on a $T$-scale much longer than unity, the term on the left hand side is negligible, and $\delta a$ is said to adiabatically follow the phase perturbations. The term in $\partial_X^2 \delta a$ will lead to phase derivatives that are higher than second order, and so can be ignored. Hence

$$a_K \delta a \simeq -K \partial_X \theta.$$

# Derivation of Phase Equation from Amplitude Equation II

## Real part

$$\partial_T \delta a = -2a_K^2 \delta a - 2K a_K \partial_X \theta + \partial_X^2 \delta a$$

For time variations on a $T$-scale much longer than unity, the term on the left hand side is negligible, and $\delta a$ is said to adiabatically follow the phase perturbations. The term in $\partial_X^2 \delta a$ will lead to phase derivatives that are higher than second order, and so can be ignored. Hence

$$a_K \delta a \simeq -K \partial_X \theta.$$

## Imaginary part

$$a_K \partial_T \theta \simeq 2K \partial_X \delta a + a_K \partial_X^2 \theta + a_K K \partial_Y^2 \theta.$$

Eliminating $\delta a$ and using $a_K^2 = 1 - K^2$ gives

$$\partial_T \theta = \left[ \frac{1 - 3K^2}{1 - K^2} \right] \partial_X^2 \theta + K \partial_Y^2 \theta.$$

the phase *diffusion equation* in scaled units.

# Derivation of Phase Equation from Amplitude Equation III

Returning to the unscaled units we get the phase diffusion equation for a phase variation $\theta = kx + \delta\theta$

$$\partial_t \theta = D_\parallel \partial_x^2 \theta + D_\perp \partial_y^2 \theta$$

with diffusion constants for the state with wave number $q = q_c + k$ (with $k$ related to $K$ by $k = \xi_0^{-1} \varepsilon^{1/2} K$)

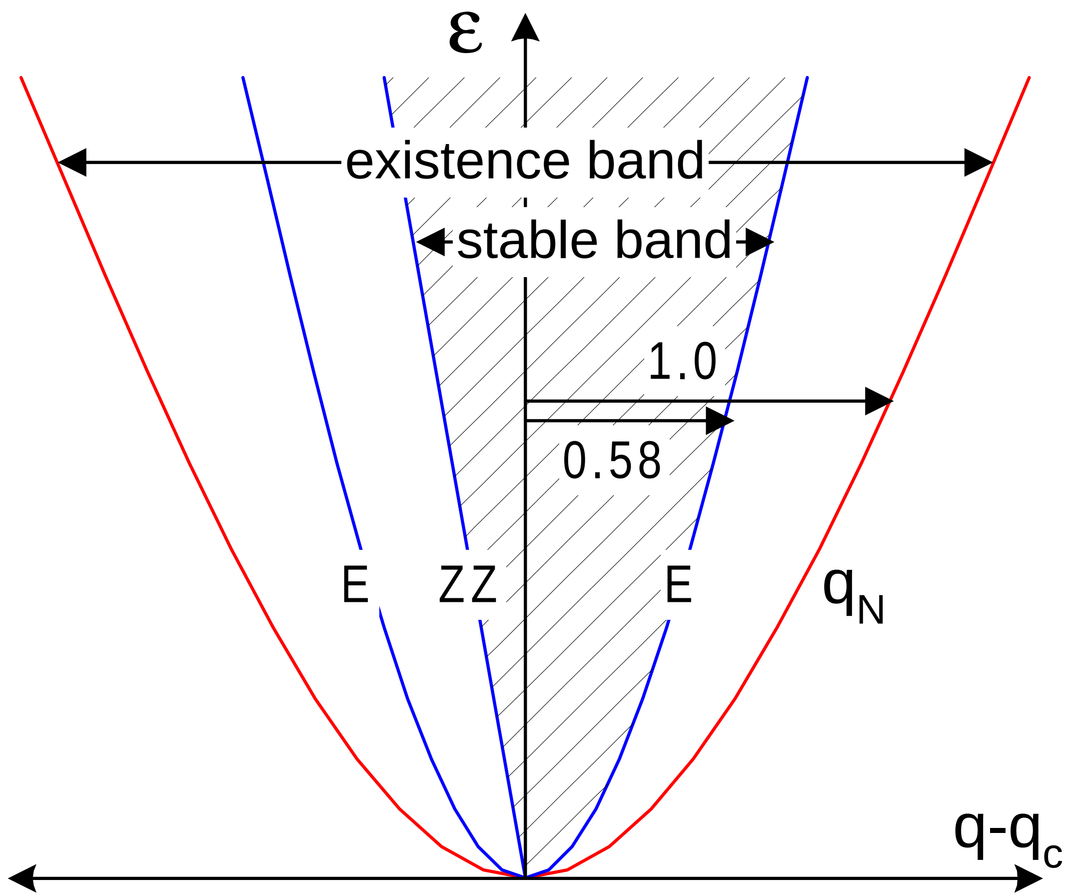
$$D_\parallel = (\xi_0^2 \tau_0^{-1}) \frac{\varepsilon - 3\xi_0^2 k^2}{\varepsilon - \xi_0^2 k^2}$$

$$D_\perp = (\xi_0^2 \tau_0^{-1}) \frac{k}{q_c}.$$

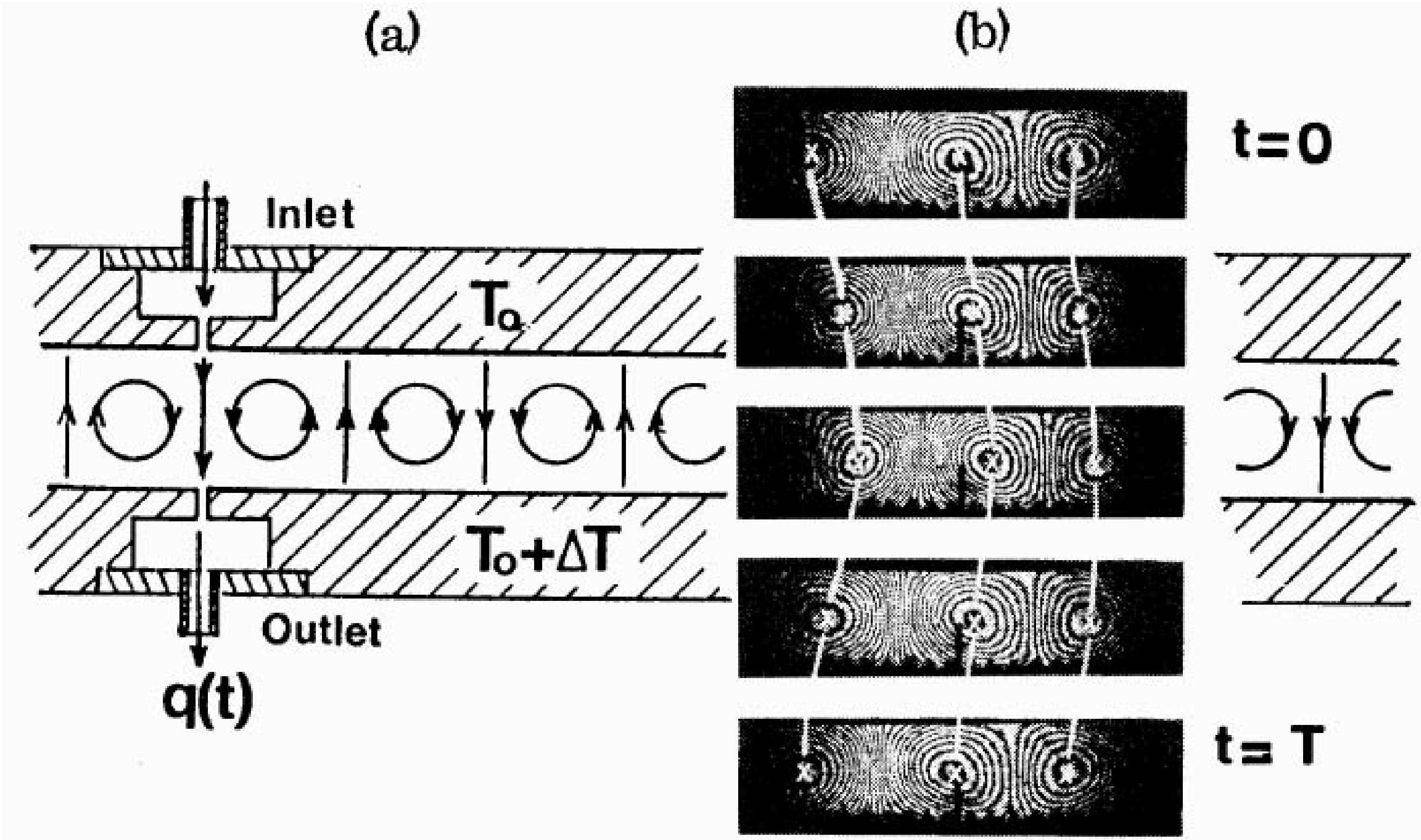
A negative diffusion constant leads to exponentially growing solutions, i.e. the state with wave number $q_c + k$ is *unstable* to long wavelength phase perturbations for

| | |
|---|---|
| $\lvert\xi_0 k\rvert > \varepsilon^{1/2}/\sqrt{3}$ | $D_\parallel < 0$: longitudinal (Eckhaus) |
| $k < 0$ | $D_\perp < 0$: transverse (ZigZag) |

# Stability Balloon Near Threshold

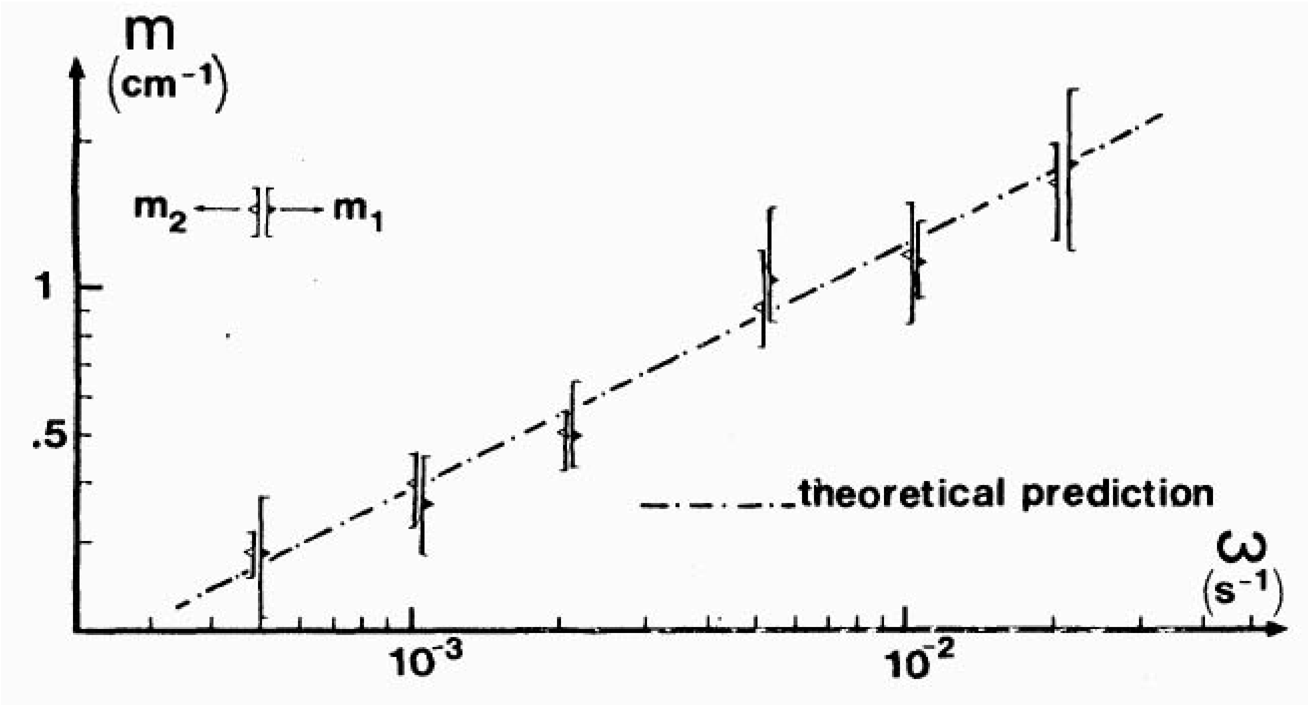


# Experimental Test of Phase Diffusion




Wesfried and Croquette (1980)

# Experimental Test of Phase Diffusion

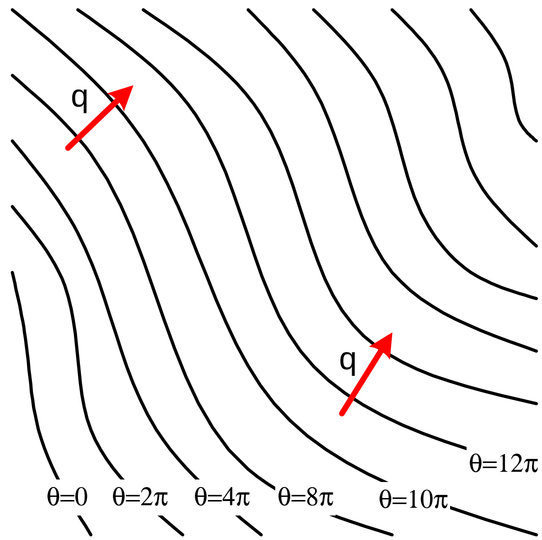



Solution to phase equation with periodic driving at frequency $\omega$

$$\theta(x,t) = \theta_0 e^{-m_1|x|}\cos(m_2|x| - \omega t) \text{ with } m_1 = m_2 = \sqrt{\omega/2D_\parallel}$$

## Phase Dynamics Away From Threshold (MCC and Newell, 1984)

Away from threshold the other degrees of freedom relax even more quickly, and so idea of a slow phase equation remains.




- pattern is given by the lines of constant phase $\theta$ of a local stripe solution;
- wave vector $\mathbf{q}$ is the gradient of this phase $\mathbf{q} = \nabla\theta$.

A nonlinear saturated straight-stripe solution with wave vector $\mathbf{q} = q\hat{\mathbf{x}}$ is

$$\mathbf{u} = \mathbf{u}_q(\theta, z, t) \qquad \theta = qx$$

A nonlinear saturated straight-stripe solution with wave vector $\mathbf{q} = q\hat{\mathbf{x}}$ is

$$\mathbf{u} = \mathbf{u}_q(\theta, z, t) \qquad \theta = qx$$

For slow spatial variations of the wave vector over a length scale $\eta^{-1}$ this leads to the ansatz for a pattern of slowly varying stripes

$$\mathbf{u} \approx \mathbf{u}_q(\theta, z, t) + O(\eta), \qquad \mathbf{q} = \nabla\theta(\mathbf{x})$$

where $\mathbf{q} = \mathbf{q}(\eta\mathbf{x})$ so that $\nabla\mathbf{q} = O(\eta)$.

A nonlinear saturated straight-stripe solution with wave vector $\mathbf{q} = q\hat{\mathbf{x}}$ is

$$\mathbf{u} = \mathbf{u}_q(\theta, z, t) \qquad \theta = qx$$

For slow spatial variations of the wave vector over a length scale $\eta^{-1}$ this leads to the ansatz for a pattern of slowly varying stripes

$$\mathbf{u} \approx \mathbf{u}_q(\theta, z, t) + O(\eta), \qquad \mathbf{q} = \nabla\theta(\mathbf{x})$$

where $\mathbf{q} = \mathbf{q}(\eta\mathbf{x})$ so that $\nabla\mathbf{q} = O(\eta)$.

We can develop an equation for the phase variation by expanding in $\eta$

$$\tau(q)\partial_t\theta = -\nabla \cdot [\mathbf{q}B(q)]$$

A nonlinear saturated straight-stripe solution with wave vector $\mathbf{q} = q\hat{\mathbf{x}}$ is

$$\mathbf{u} = \mathbf{u}_q(\theta, z, t) \qquad \theta = qx$$

For slow spatial variations of the wave vector over a length scale $\eta^{-1}$ this leads to the ansatz for a pattern of slowly varying stripes

$$\mathbf{u} \approx \mathbf{u}_q(\theta, z, t) + O(\eta), \qquad \mathbf{q} = \nabla\theta(\mathbf{x})$$

where $\mathbf{q} = \mathbf{q}(\eta\mathbf{x})$ so that $\nabla\mathbf{q} = O(\eta)$.

We can develop an equation for the phase variation by expanding in $\eta$

$$\tau(q)\partial_t\theta = -\nabla\cdot[\mathbf{q}B(q)]$$

The form of the equation derives from symmetry and smoothness arguments, and expanding up to second order derivatives of the phase.

The parameters $\tau(q)$, $B(q)$ are system dependent functions depending on the equations of motion, $\mathbf{u}_q$, etc.

# Small Deviations from Stripes

$$\tau(q)\partial_t\theta = -\nabla\cdot[\mathbf{q}B(q)]$$

For $\theta = qx + \delta\theta$ this reduces to

$$\partial_t\delta\theta = D_\parallel(q)\partial_x^2\delta\theta + D_\perp(q)\partial_y^2\delta\theta$$

with

$$D_\perp(q) = -\frac{B(q)}{\tau(q)}$$

$$D_\parallel(q) = -\frac{1}{\tau(q)}\frac{d(qB(q))}{dq}$$

# Small Deviations from Stripes

$$\tau(q)\partial_t\theta = -\nabla \cdot [\mathbf{q}B(q)]$$

For $\theta = qx + \delta\theta$ this reduces to

$$\partial_t\delta\theta = D_\parallel(q)\partial_x^2\delta\theta + D_\perp(q)\partial_y^2\delta\theta$$

with

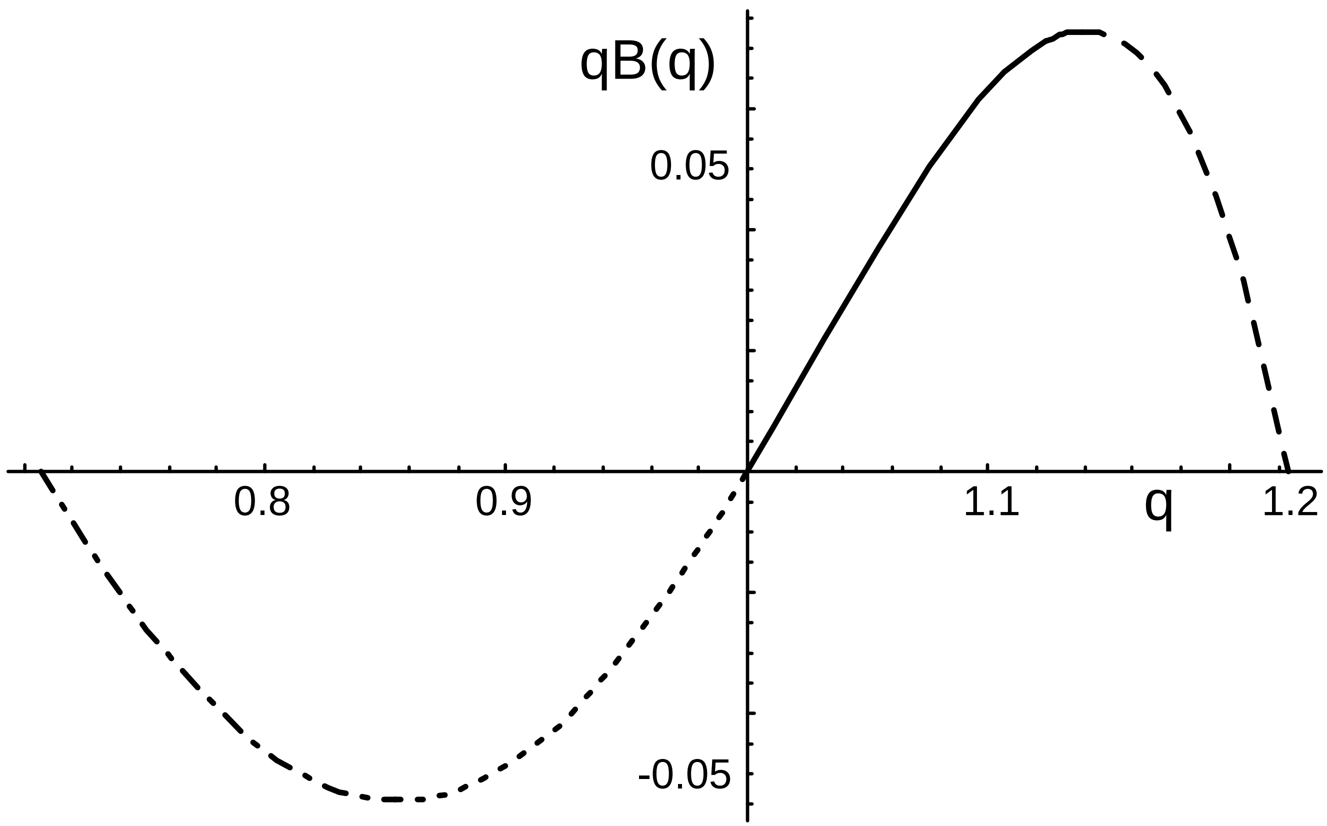
$$D_\perp(q) = -\frac{B(q)}{\tau(q)}$$

$$D_\parallel(q) = -\frac{1}{\tau(q)}\frac{d(qB(q))}{dq}$$

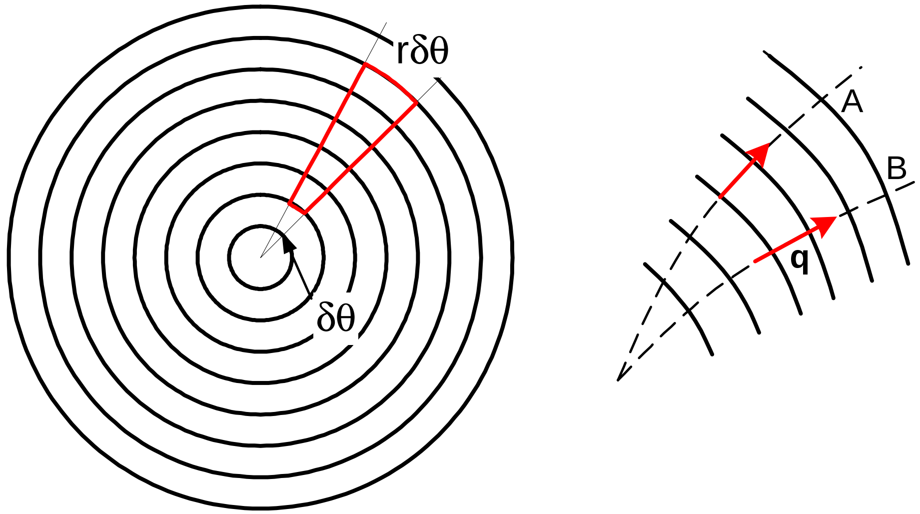
A negative diffusion constant signals instability:

- $[qB(q)]' < 0$: Eckhaus instability
- $B(q) < 0$: zigzag instability

# Phase Parameters for the Swift-Hohenberg Equation



# Application: Wave Number Selection by a Focus

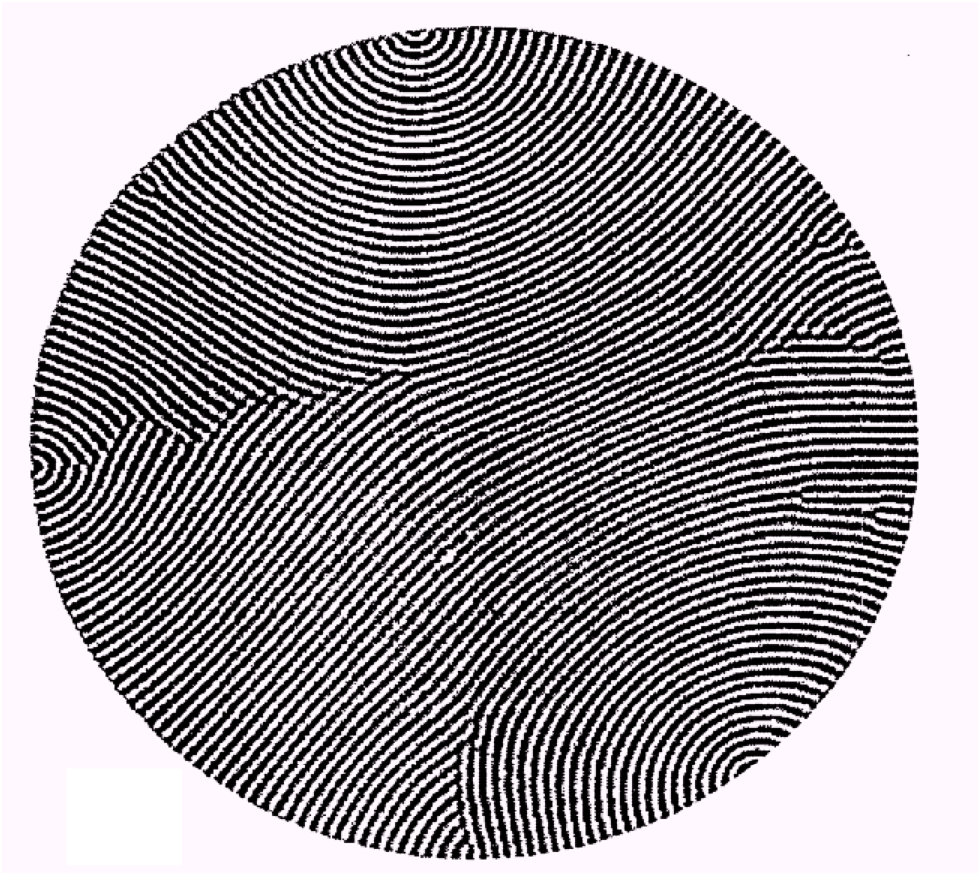



$$\nabla \cdot (\mathbf{q} B(q)) = 0 \qquad \Rightarrow \qquad \oint B(q)\mathbf{q} \cdot \hat{\mathbf{n}}\, dl = \mathbf{0}$$

$$q B(q) = \frac{C}{r} \underset{r \to \infty}{\longrightarrow} 0$$

i.e. $q \to q_f$ with $B(q_f) = 0$, the wave number of the zigzag instability!

# Defects



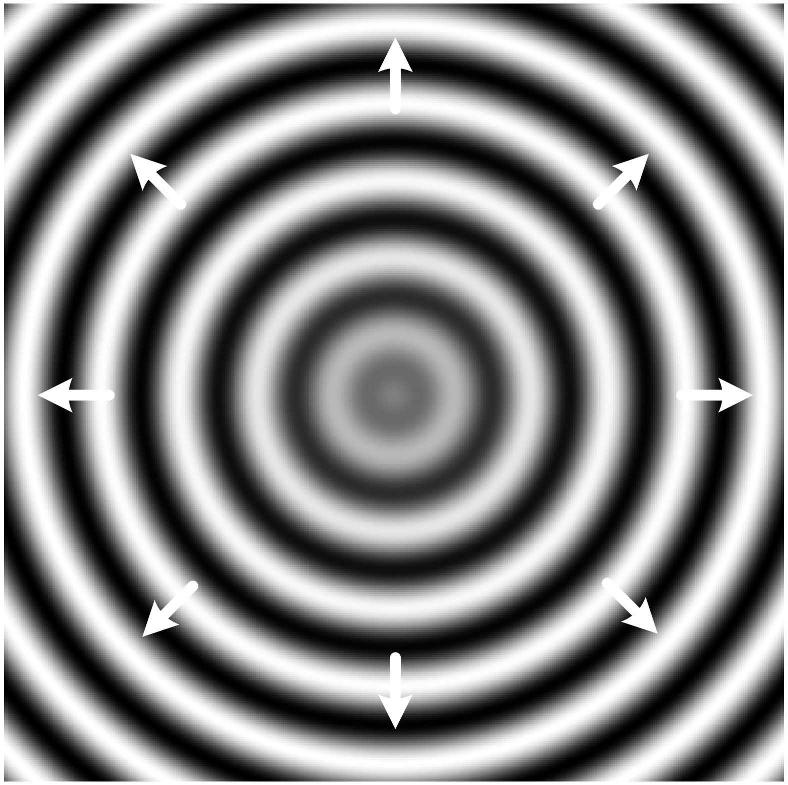
# Defects and Broken Symmetries

There are *two* broken symmetries: rotational and translational. Correspondingly there are two types of topological defects:

- Rotational (associated with direction of wavevector **q**)
  - ◇ Focus, disclination (point defect)
  - ◇ Grain boundary (line defect)
- Translational (associated with phase $\theta$)
  - ◇ Dislocation (point defect)

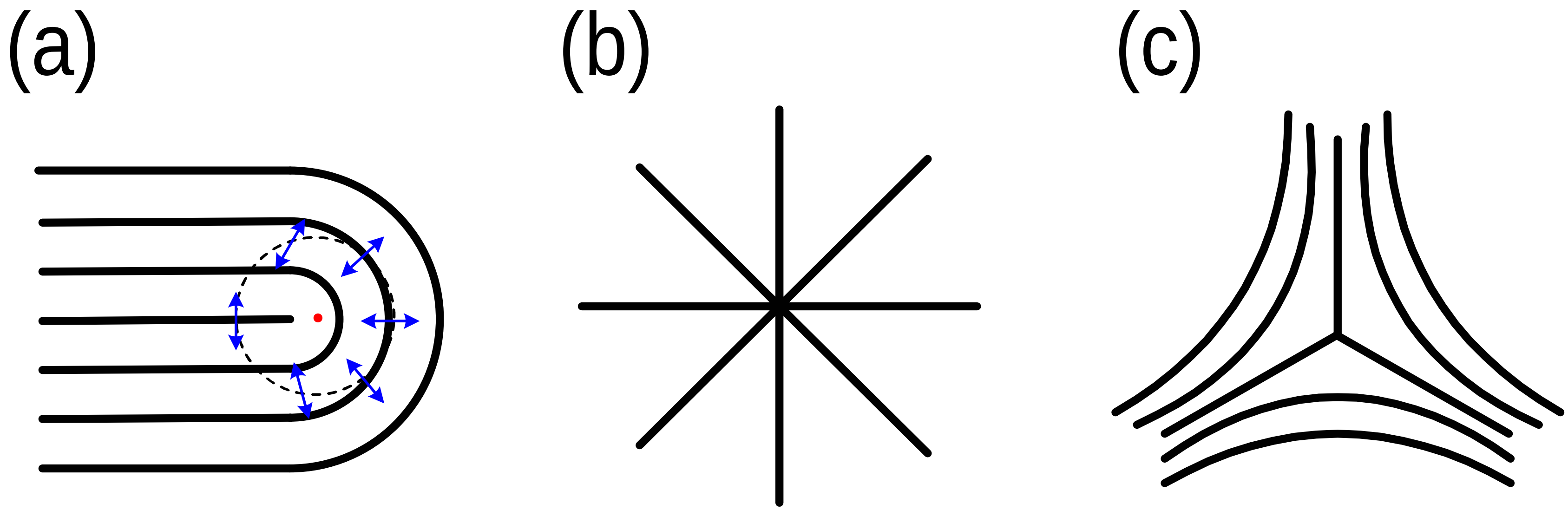
Unfortunately the rotational defects cannot be considered independently of the broken translational symmetry, and this complicates the discussion.

# Focus/target defect



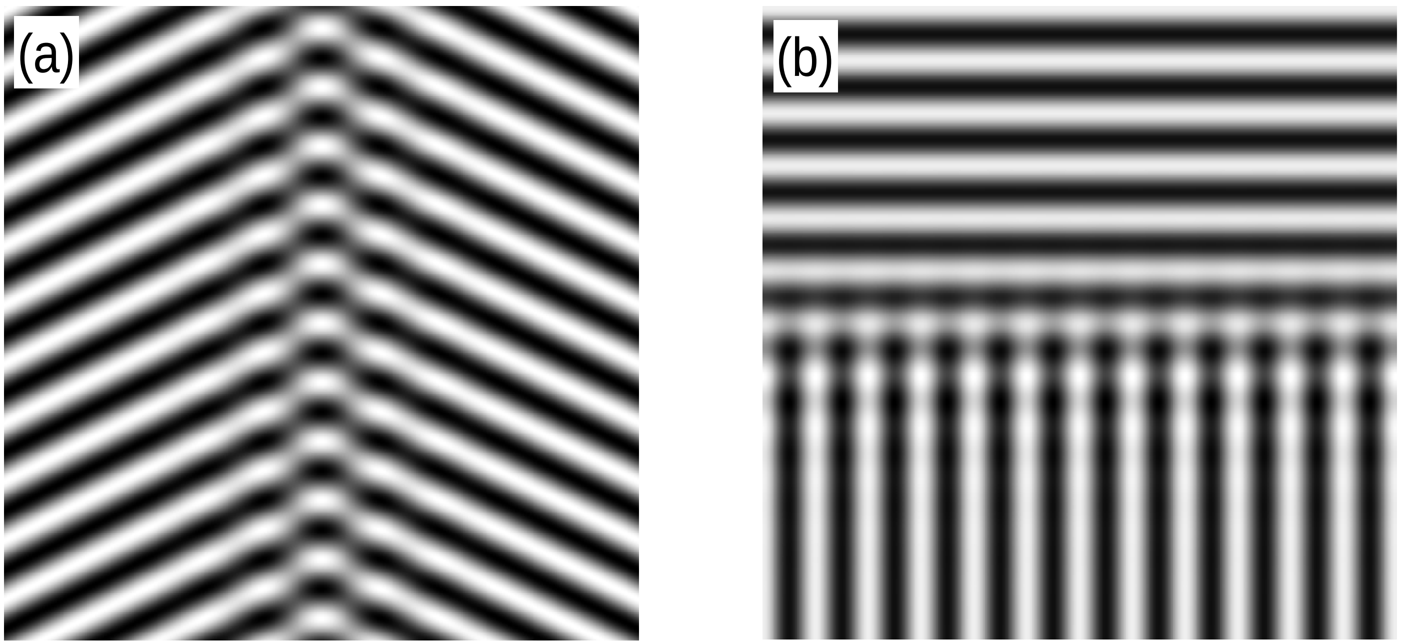
Wavevector winding number = 1

# Disclinations

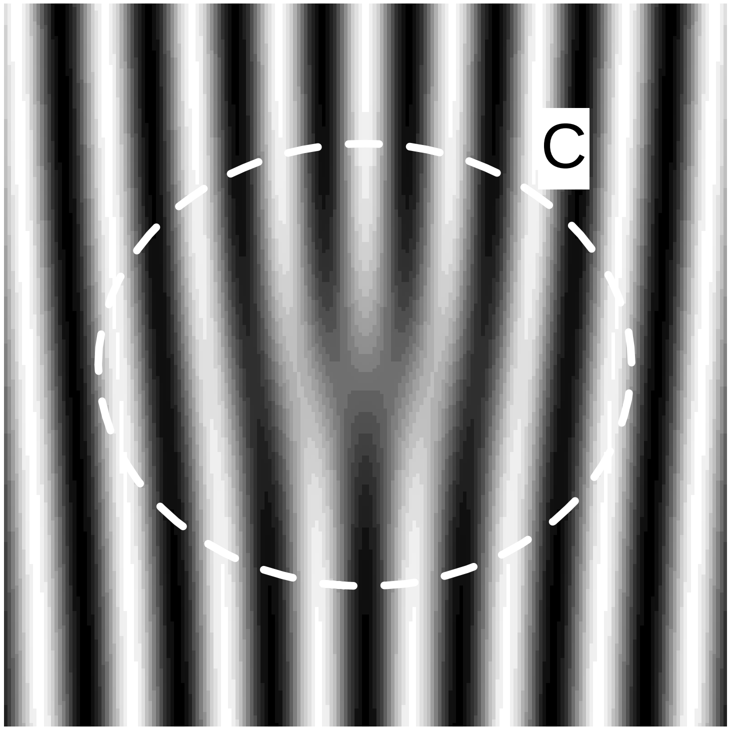



Winding numbers: (a) $\frac{1}{2}$; (b) 1; (c) -1

# Grain Boundaries

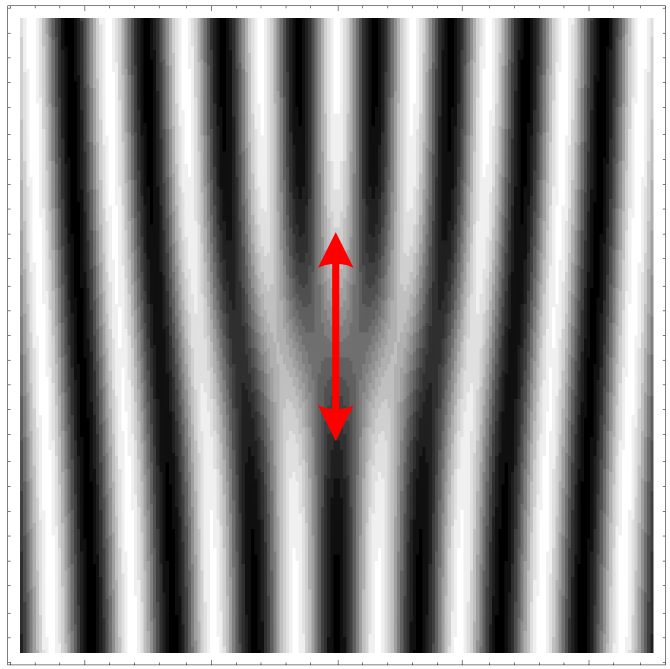


# Dislocation



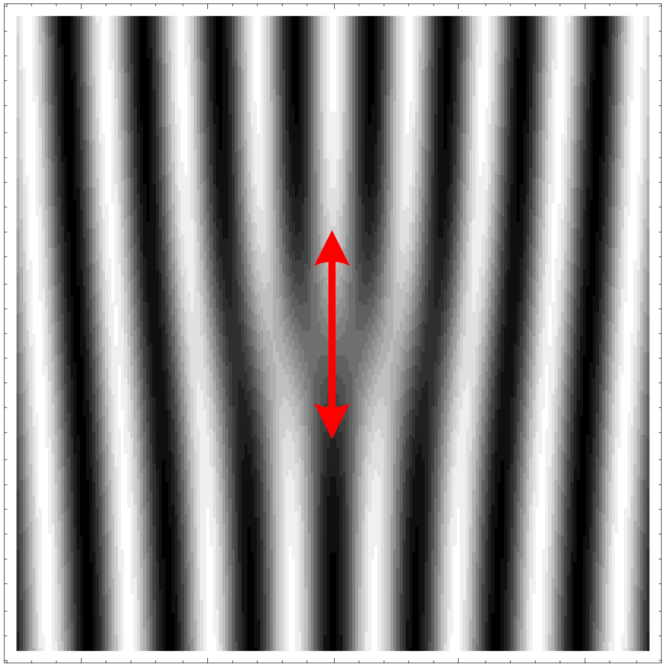

$$\text{Phase winding number } = \frac{1}{2\pi}\oint \nabla\theta \cdot \mathbf{dl} = 1$$

# Dislocation Climb



Climb motion is through symmetry related states and is smooth

# Dislocation Climb

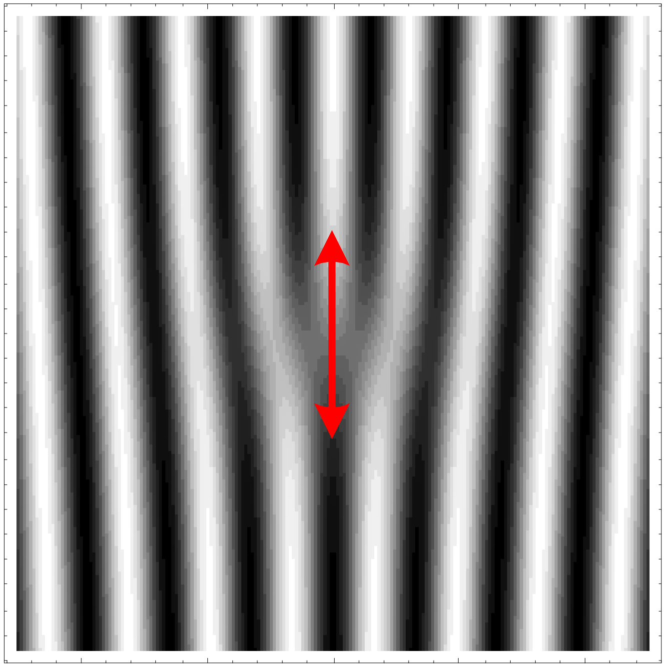


Climb motion is through symmetry related states and is smooth

Climb velocity

$$v_d \approx \beta(q - q_d)$$

# Dislocation Climb



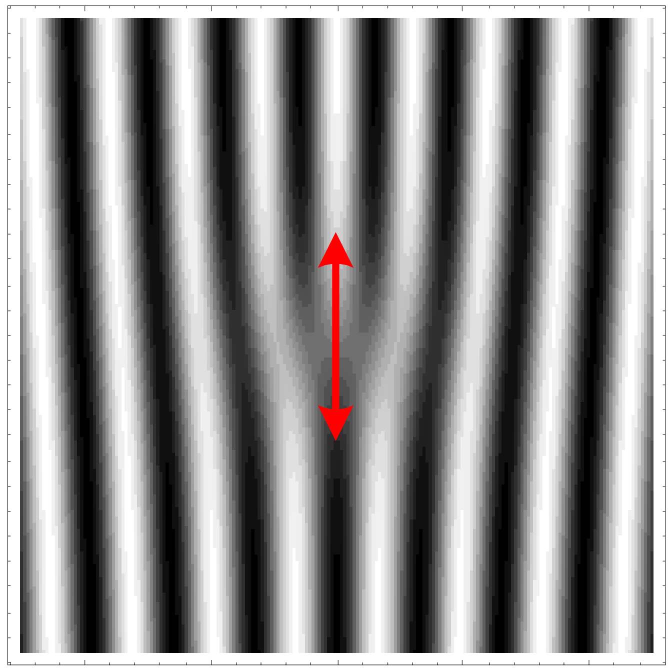
Climb motion is through symmetry related states and is smooth

Climb velocity

$$v_d \approx \beta(q - q_d)$$

What is $q_d$?

# Dislocation Climb



Climb motion is through symmetry related states and is smooth

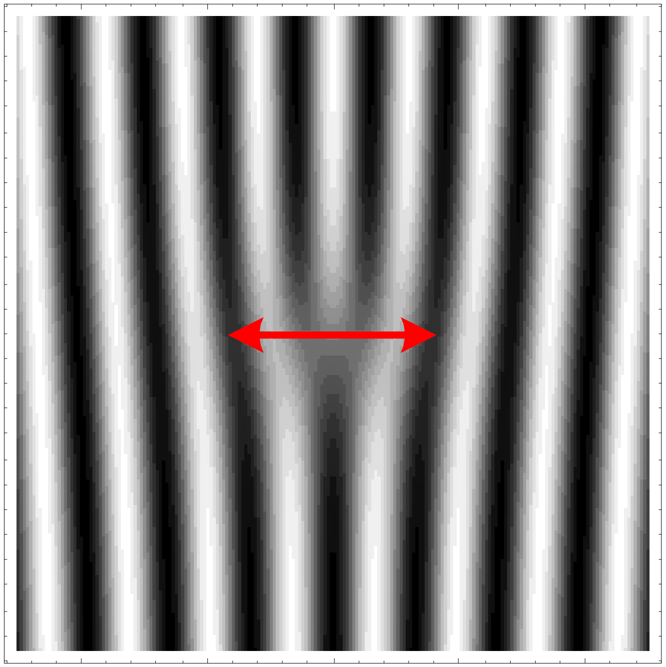
Climb velocity

$$v_d \approx \beta(q - q_d)$$

What is $q_d$? Easy in equilibrium systems, not in ones far from equilibrium.

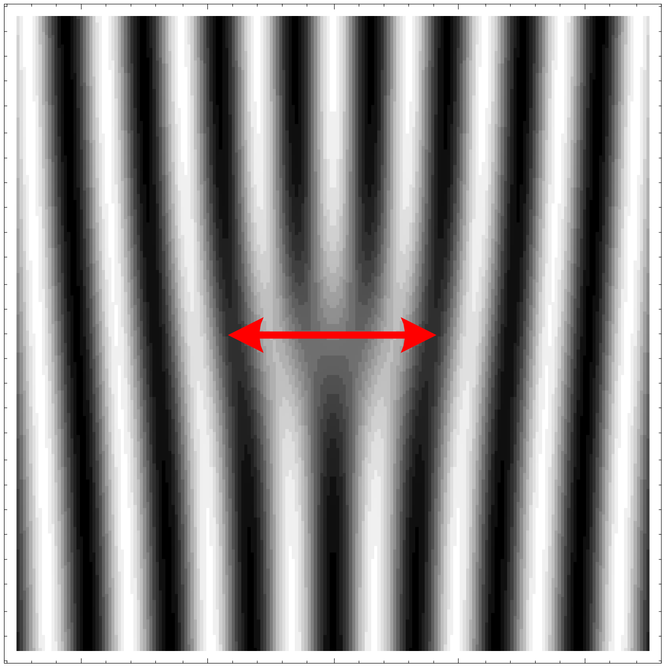
Convection experiments (from website of Eberhard Bodenschatz)

# Dislocation Glide



Glide motion involves stripe pinch off, and is pinned to the periodic structure

# Dislocation Glide



Glide motion involves stripe pinch off, and is pinned to the periodic structure

Dislocation motion is important in the relaxation of patterns.

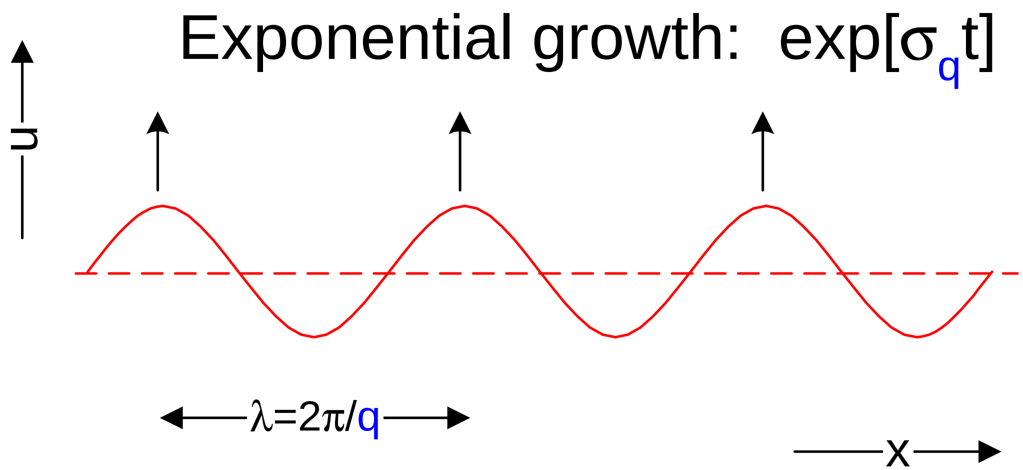
## Oscillatory Instabilities

The same set of ideas can be applied to oscillatory instabilities
$\mathrm{Im}\,\sigma(\mathbf{q} = \mathbf{q}_c) \neq 0$

- Now the amplitude equation is the Complex Ginzburg-Landau (CGL) equation (complex coefficients)
- Phase equation can be used to understand shocks

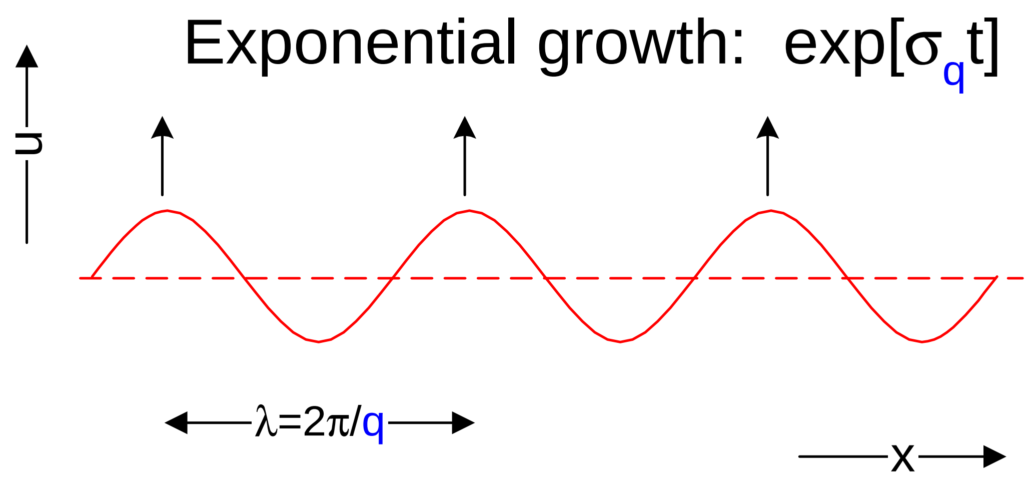
I will briefly discuss the case of an instability to spatially uniform oscillations ($q_c = 0$)

# Linear Instability




$$\delta \mathbf{u}_{\mathbf{q}}(\mathbf{x}_\perp, z, t) = \mathbf{u}_{\mathbf{q}}(z)\, e^{i\,\mathbf{q}\cdot\mathbf{x}_\perp}\, e^{\sigma_{\mathbf{q}} t}$$

# Linear Instability

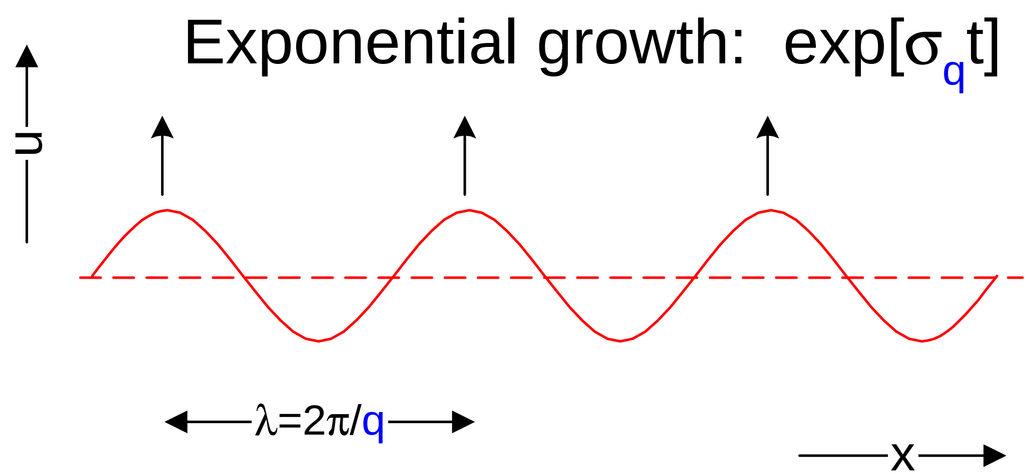



$$\delta \mathbf{u_q}(\mathbf{x}_\perp, z, t) = \mathbf{u_q}(z)\, e^{i\,\mathbf{q}\cdot\mathbf{x}_\perp}\, e^{\sigma_\mathbf{q} t}$$

If $\omega_c = -\operatorname{Im}\sigma_{q_c} \neq 0$ we have an instability to

- for $q_c = 0$: a nonlinear oscillator which also supports travelling waves
- for $q_c \neq 0$: a wave pattern (standing or travelling)

# Linear Instability



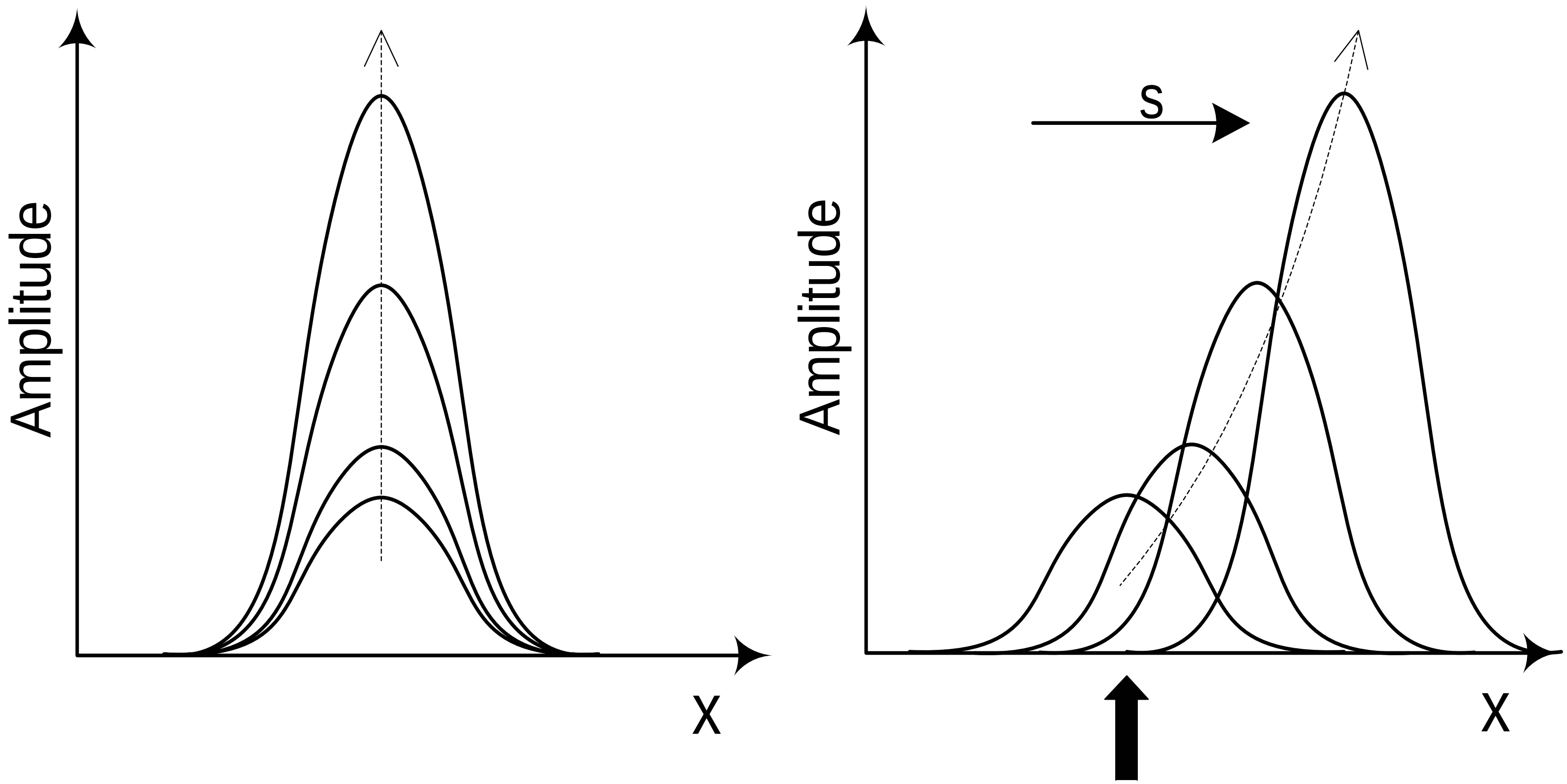

$$\delta \mathbf{u}_{\mathbf{q}}(\mathbf{x}_\perp, z, t) = \mathbf{u}_{\mathbf{q}}(z)\, e^{i\,\mathbf{q}\cdot\mathbf{x}_\perp}\, e^{\sigma_{\mathbf{q}} t}$$

If $\omega_c = -\operatorname{Im}\sigma_{q_c} \neq 0$ we have an instability to

- for $q_c = 0$: a nonlinear oscillator which also supports travelling waves
- for $q_c \neq 0$: a wave pattern (standing or travelling)

Important new concept: absolute v. convective instability

# Absolute and Convective Instability



# Conditions for Convective and Absolute Instability

- **Convective instability:** same as condition for instability to Fourier mode

$$\mathrm{Max}_{\mathbf{q}}\ \mathrm{Re}\,\sigma(\mathbf{q}) = 0$$

- **Absolute instability:** for a growth rate spectrum $\sigma_q$, the system is absolutely unstable if

$$\mathrm{Re}\,\sigma(\mathbf{q}_s) = 0$$

where $\mathbf{q}_s$ is a *complex* wave vector given by the solution of the stationary phase condition

$$\frac{d\sigma_{\mathbf{q}}}{d\mathbf{q}} = 0$$

# Derivation of Condition for Absolute Instability

In the linear regime the disturbance growing from any given initial condition $u_p(\mathbf{x}, t = 0)$ can be expressed as

$$u_p(x,t) = \int_{-\infty}^{\infty} dq\, e^{iqx+\sigma_q t} \int_{-\infty}^{\infty} dx'\, u_p(x',0)e^{-iqx'}$$

Rewrite the integral as

$$u_p(x,t) = \int_{-\infty}^{\infty} dx' u_p(x',0) \int_{-\infty}^{\infty} dq\, e^{iq(x-x')+\sigma_q t}$$

For large time and at fixed distance the integral can be estimated using the stationary phase method: the integral is dominated by the region around the complex wave number $q = q_s$ given by the solution of

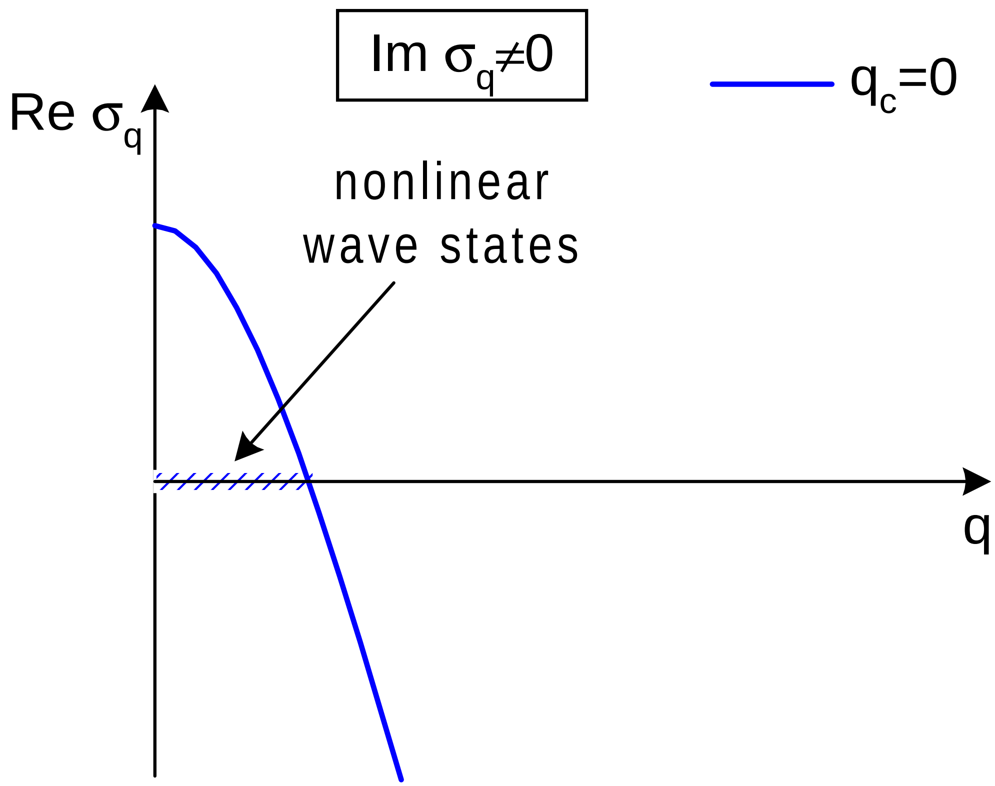
$$\frac{d\sigma_q}{dq} = 0$$

Estimating the integral from the value of the integrand at the stationary phase point gives

$$u_p(x = 0, t) \sim e^{\sigma_{q_s} t}$$

Thus the system will be absolutely unstable for $\mathrm{Re}\,\sigma_{q_s} > 0$.

# Oscillatory Intability: Complex Ginzburg-Landau




1d: $$\partial_T \bar{A} = (1 + ic_0)\bar{A} + (1 + ic_1)\partial_X^2 \bar{A} - (1 - ic_3)\left|\bar{A}\right|^2 \bar{A}$$

2d: $$\partial_T \bar{A} = (1 + ic_0)\bar{A} + (1 + ic_1)\nabla_\perp^2 \bar{A} - (1 - ic_3)\left|\bar{A}\right|^2 \bar{A}$$

# Simulations of the CGL Equation

General equation (2d)

$$\partial_T \bar{A} = (1 + ic_0)\bar{A} + (1 + ic_1)\nabla_\perp^2 \bar{A} - (1 - ic_3)\left|\bar{A}\right|^2 \bar{A}$$

Case simulated:

- $c_0 = -c_3$ (no loss of generality) for simplicity of plots
- $c_1 = 0$ (choice of parameters)

$$\partial_T \bar{A} = (1 - ic_3)\bar{A} + \nabla_\perp^2 \bar{A} - (1 - ic_3)\left|\bar{A}\right|^2 \bar{A}$$

Simulations…

# Nonlinear Wave Patterns

As well as uniform oscillations, CGL equation supports travelling wave solutions, but with properties that are strange to those of us brought up on linear waves:

# Nonlinear Wave Patterns

As well as uniform oscillations, CGL equation supports travelling wave solutions, but with properties that are strange to those of us brought up on linear waves:

- Waves annihilate at shocks rather than superimpose
- Waves disappear at boundaries rather than reflect (not shown)
- Defects: importance as persistent sources
- Spiral defects play a conspicuous role, because they are topologically defined persistent sources
- Shocks between spiral defects lead to exponential decay of interaction (not $1/r$ as in real amplitude equation)
- Instabilities can lead to spatiotemporal chaos

# Wave Solutions

$$\partial_T \bar{A} = (1 + ic_0)\bar{A} + (1 + ic_1)\nabla_\perp^2 \bar{A} - (1 - ic_3)\left|\bar{A}\right|^2 \bar{A}$$

Travelling wave solutions

$$\bar{A}_K(\mathbf{X}, T) = a_K e^{i(\mathbf{K}\cdot\mathbf{X} - \Omega_K T)}$$

$$a_K^2 = 1 - K^2 \qquad \Omega_K = -(c_0 + c_3) + (c_1 + c_3)K^2$$

Group speed

$$S = d\Omega_K/dK = 2(c_1 + c_3)K$$

Standing waves, based on the addition of waves at $\mathbf{K}$ and $-\mathbf{K}$ can be constructed, but they are unstable towards travelling waves

## Stability Analysis

$$\bar{A}_K(\mathbf{X}, T) = (a_K + \delta a)e^{i(\mathbf{K}\cdot\mathbf{X} - \Omega_K T + \delta\theta)}$$

For *small, slowly varying* phase perturbations

$$\partial_T \delta\theta + S\partial_X \delta\theta = D_\parallel(K)\partial_X^2 \delta\theta + D_\perp(K)\partial_Y^2 \delta\theta$$

with longitudinal and transverse diffusion with constants

$$D_\parallel(K) = (1 - c_1c_3)\frac{1 - \nu K^2}{1 - K^2} \qquad D_\perp(K) = (1 - c_1c_3)$$

with

$$\nu = \frac{3 - c_1c_3 + 2c_3^2}{1 - c_1c_3}$$

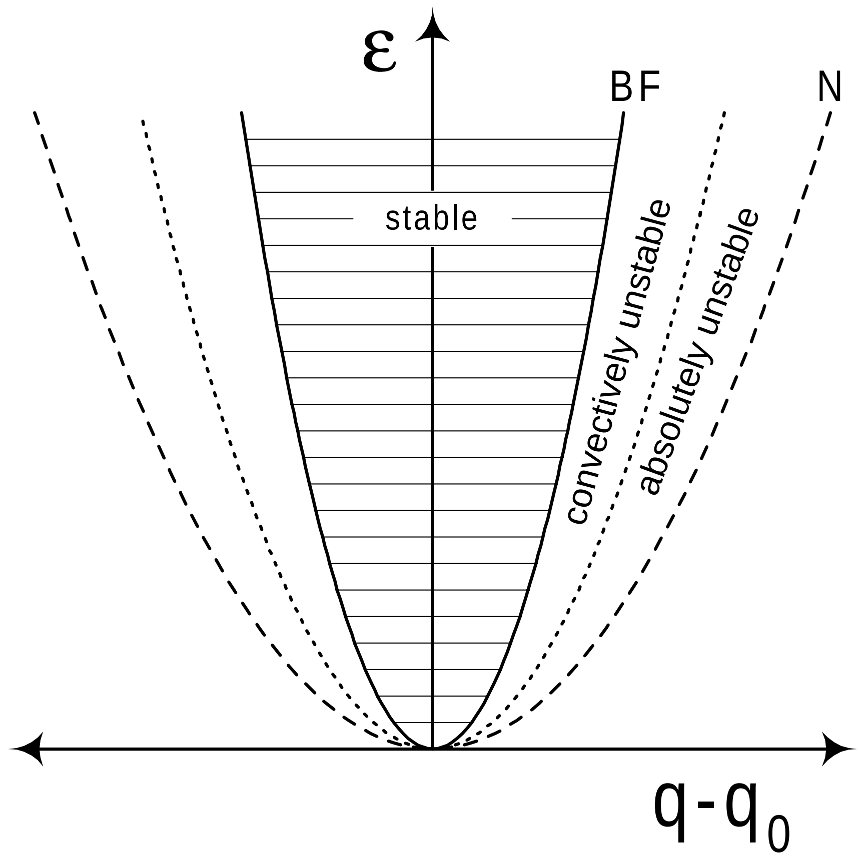
- $D_\parallel = 0 \Rightarrow$ Benjamin-Feir instability (longitudinal sideband instability analogous to Eckhaus) for

$$|K| \geq \Lambda_B = \nu^{-1}$$

  leaving a stable band of wave numbers with width a fraction $\nu^{-1}$ of the existence band.

- For $1 - c_1c_3 < 0$ *all* wave states are unstable (Newell)

# Stability Balloon



# Nonlinear Phase Equation

For slow phase variations about spatially uniform oscillations (now keeping all terms up to second order in derivatives)

$$\partial_T \theta = \Omega + \alpha \nabla_\perp^2 \theta - \beta (\vec{\nabla}_\perp \theta)^2$$

with

$$\alpha = 1 - c_1 c_3$$
$$\beta = c_1 + c_3$$
$$\Omega = c_0 + c_3$$

Can be used to understand shocks

# Cole-Hopf Transformation

The Cole-Hopf transformation

$$\chi(X,Y,T) = \exp[-\beta\theta(X,Y,T)/\alpha]$$

transforms the nonlinear phase equation into the *linear* equation for $\chi$

$$\partial_T \chi = \alpha \nabla_X^2 \chi$$

Plane wave solutions

$$\chi = \exp\left[(\mp\beta K X + \beta^2 K^2 T)/\alpha\right]$$

correspond to the phase variations

$$\theta = \pm K X - \beta K^2 T$$

## Cole-Hopf Transformation (cont)

Since the $\chi$ equation is *linear*, we can superimpose a pair of these solutions

$$\chi = \exp\left[(-\beta K X + \beta^2 K^2 T)/\alpha\right] + \exp\left[(+\beta K X + \beta^2 K^2 T)/\alpha\right]$$

The phase is

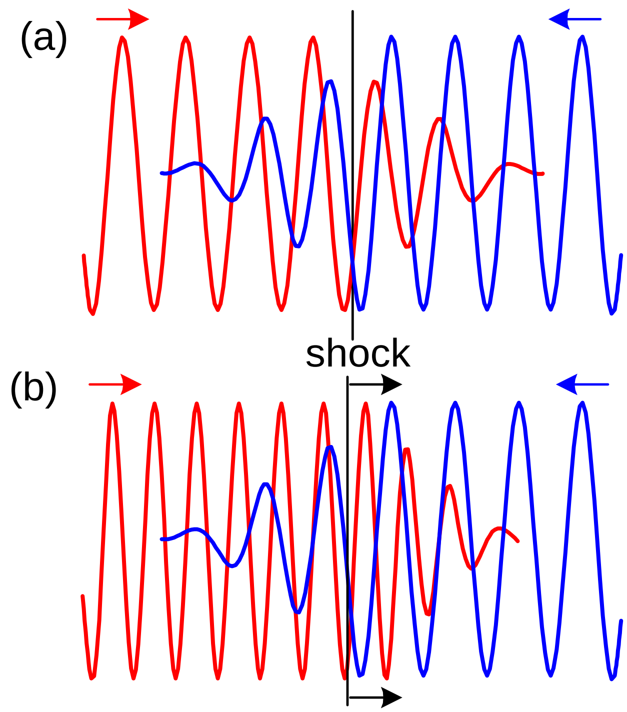
$$\theta = -\beta K^2 T - \frac{\alpha}{\beta} \ln[2\cosh(\beta K X/\alpha)].$$

For large $|X|$ the phase is given by (assuming $\beta K$ positive)

$$\theta \to -K X - \beta K^2 T - \tfrac{\alpha}{\beta} \exp(-2\beta K X/\alpha) \quad \text{for} \quad X \to +\infty$$

i.e. left moving waves plus exponentially decaying right moving waves with the decay length $\alpha/2\beta K$. Similarly for $X \to -\infty$ get left moving waves with exponentially small right moving waves.

# Shocks




- Shocks are sinks, not sources
- For positive group speed shocks between waves of different frequency move so that the higher frequency region expands

# Spiral Defects

m-armed spiral: $$\oint \nabla\theta \cdot \mathbf{dl} = m \times 2\pi$$

$$\bar{A} = a(R)e^{i(K(R)R+m\theta-\Omega_s T)}$$

with for $R \to \infty$

$$a(R) \to a_K \qquad K(R) \to K_s \qquad \text{with} \qquad \Omega_K(K_s) = \Omega_s$$

## Uniqueness

A key question is whether there is a family of spirals giving a continuous range of possible frequencies $\Omega_s$, or there is a unique spiral structure with a prescribed frequency that selects a particular wave number (or possibly a discrete set of possible spirals).

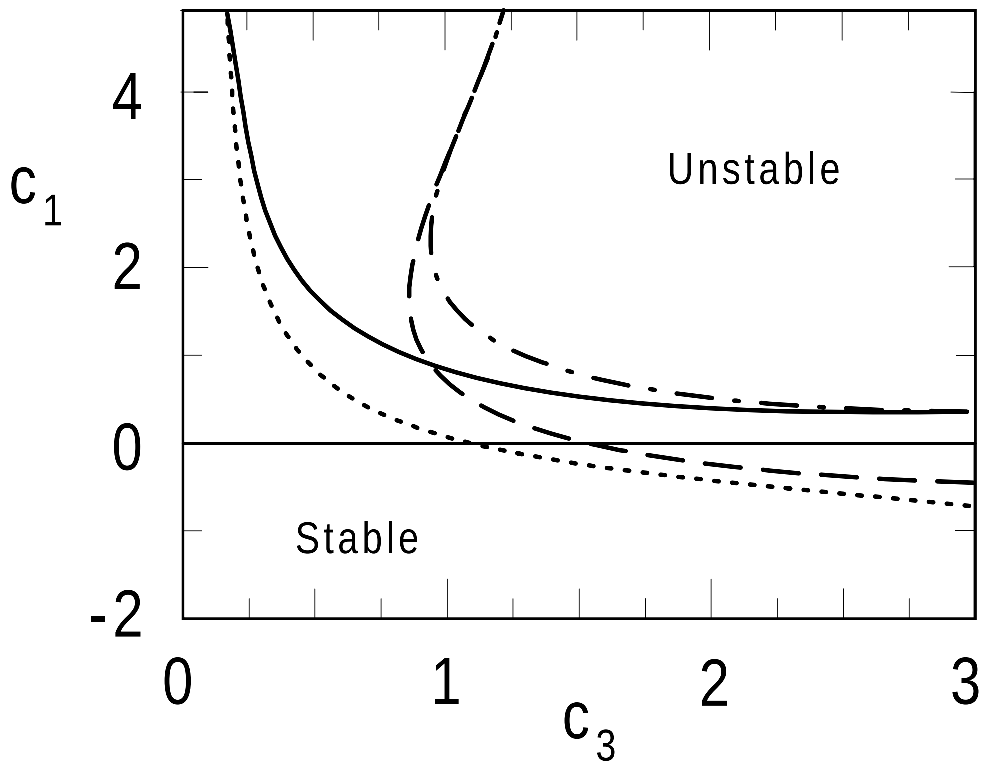
A perturbative treatments of the CGLE for small $c_1 + c_3$ about the real amplitude equation predicts a unique stable spiral structure, with a wave number $K_s$ that varies as (Hagan, 1982)

$$K_s \to \frac{1.018}{|c_1 + c_3|} \exp[-\frac{\pi}{2\,|c_1 + c_3|}].$$

## Stability Revisited

- Wave number of nonlinear waves determined by spirals
- Only BF stability of waves at $K_s$ relevant to stability of periodic state
- Convective instability may not lead to breakdown
- Core instabilities may intervene

Stability lines of the CGLE (unstable states are towards larger positive $c_1 c_3$)




solid line: Newell criterion $c_1 c_3 = 1$

dotted line: convective Benjamin-Feir instability of spiral-selected wavenumber

dashed line: absolute instability of spiral selected wavenumber

dashed-dotted line: absolute instability of whole wavenumber band

## Waves in Excitable Media

Waves in reaction-diffusion systems such as chemicals or heart tissue show similar properties to waves in the CGL

[From Winfree and Strogatz (1983) and the website of G. Bub, McGill]